International Journal of Computer and Information Technology (ISSN: 2279 – 0764)
Volume 04 – Issue 03, May 2015

A New Improvement of Shishkin Fitted Mesh Technique on a First Order Finite Difference Method

With applications on singularly perturbed boundary value problem ODE

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***Abstract*— We consider a class of linear singular perturbation ODE-BVP associated with, the common Dirichlet boundary condition, as one of the most comprehensive and difficult boundary conditions which with boundary layers at end points or interior layer. A new improvement of general form of Shishkin piecewise uniform fitted mesh technique applied, of adjustable width. Taking into consideration the locating process to find, the true locations of the fine subintervals in which corresponding to singular boundary layers (viscous parts) or interior layers, occur in their solutions using some well known easy analytical and/or applied techniques, then the fine subinterval also divided to two other a little bit different distance subintervals because of the different rate of convection and diffusion phenomenon in the solutions, as alternative of the well known uniform or equidistant mesh for backward finite difference method, to introduce what a known by a fitted mesh. So 9 standard problems solved and then compared with versus numerical solution of uniform mesh and Shishkin mesh inside backward finite difference method for different choices of ε and N . In addition to presenting illustrative Matlab plots of most of the mesh constructions, the solutions and the epsilon convergence for each case separately in order to show the verification of progress and efficiency of the new method relative to both methods.**

Keywords- Singularly perturbed problems, Mesh generation and refinement, Finite differences method

I. INTRODUCTION

A singularly perturbed differential equation (SPDE) problem is a differential equation problem with a small parameter ε multiplying some or all of the terms involving the highest order derivatives. The physical properties associated with a solution containing a boundary layer function are reflected by the mathematical properties of the solution of (SPDE), as

$$-\varepsilon u''(x) + b(x)u'(x) + c(x)u(x) = f(x) \qquad (1a)$$
$$\text{for } x \in (0,1), u(0) = u(1) = 0, \qquad 0 < \varepsilon \ll 1. \quad (1b)$$

When $c = 0$ this is known as a convection-diffusion problem, whereas if $b = 0$ and $c \neq 0$, it is of reaction-diffusion type.

Solution $u(x,\varepsilon)$ of (1) and its derivatives approach a discontinuous limit as ε approches zero. These problems are characterized by the property that the solution has different asymptotic expansions in distinguished sub domains of the entire given domain. They present layers where the solution changes abruptly. We consider the convection-diffusion equation through imagine a river flowing, strongly and smoothly, and some ink pours into the water at a certain point will lead to two physical processes operate:

1) Convection alone would carry the ink along a one-dimensional curve on the surface. If the flow is fast, this is the dominant mechanism.

2) The ink diffuses slowly through the water, it makes curve spread out gradually.

Classical convergence theory for finite-difference method, which is method involving difference quotient approximations for derivatives can be used for solving certain second-order boundary value problems, is based on the complementary concepts of consistency and stability

consistency+stability⇒convergence.

But for singularly perturbed problems, if any discretization technique is applied, need to analyze carefully the dependence on the parameter ε of those constants that arise in consistency, stability and error estimates. Truncation error may depend on ε. The most common anomalous behavior that appears when finite-difference schemes are used is forward or backward difference operator on uniform meshes does not necessarily give satisfactory numerical solutions. Usually the pointwise error of such solutions increases as the mesh is refined, to a stage where **the mesh parameter (h) is of the same order of magnitude** as **the singular perturbation parameter ε.** One obvious requirement for a numerical method being applied to these kinds of problems is that the pointwise errors of its solutions be bounded independently of ε and that they decrease as the mesh is redefined, at the rate which should also be independent of ε. Such requirements are not special, and for problem (1) we know some results about the analytical behaviour of the solution, that justify such requirements. The maximum principle provides a simple proof of the stability inequality, showing that the solution is bounded

$$\|u\|_{\infty} \leq C\|f\|_{\infty},$$

C independent of ε, with problem (1) satisfying $b(x) \geq b_0 > 0$.

Also the maximum principle with some techniques helps us to prove that $u(x;\varepsilon)$, solution of (1), satisfies. Considering finite difference methods, two approaches have been taken, in order to construct ε-uniform numerical schemes; the first is fitted operators approach which is not our field of study in this paper and the second is fitted meshes approach by adapting meshes to the nature of the differential operator, Where the meshes are taken such that are not uniform with highly non equidistant grids or logarithmic grids as that of Bakhvalov and the convergence analysis for these schemes is, not well clarified, at the moment.

[1] ,[2] ,[3] ,[4] ,[5] ,[6]

II. USAGE AREAS

The study of singular perturbation problems is exceptionally useful because they describe the physics of many things of academic, in fluid flows like pollution, convective heat or mass transport problems and economic interest. They may be used to model the weather, ocean currents, water flow in a pipe, the air's flow around a wing, and motion of stars inside a galaxy. Equations are ubiquitous in mathematical problems in science and engineering. Examples include the Navier–Stokes equations of fluid flow at high Reynolds number, the equations governing flow in a porous medium, the drift–diffusion equations of semiconductor device physics, fluid mechanics, elasticity, quantum mechanics, plasticity, oceanography, meteorology, reaction–diffusion processes, and mathematical models of liquid materials and chemical reactions. [7], [8] [9]

III. GOAL OF MESH REFINMENT

For stiff problems, the goal of most of strategies as well as our own is the same: **the construction of a mesh on which all features of the solution are locally smooth**. [7]

IV. SURVEY OF PREVIOUS STUDIES

The fundamental paper by Courant, Friedrichs and Lewy (1928) was on the solutions of the problems of mathematical physics by means of finite differences. A finite difference approximation was first defined for the wave equation and the Courant–Friedrichs–Lewy condition (CFL condition) was shown to be necessary for convergence. Error bounds for difference approximations of elliptic problems were first derived by Gerschgorin (1930) whose work was based on a discrete analogue of the maximum principle for Laplace's equation. This approach was pursued through the 1960s and various approximations of elliptic equations and associated boundary conditions were analyzed. Independently of the engineering applications, a number of papers appeared in the mathematical literature in the mid-1960s which were concerned with the construction and analysis of finite difference schemes by the Rayleigh-Ritz procedure with piecewise linear approximating functions. The history of numerical methods for convection-diffusion problems begins about 30 years ago, in 1969. In this year, two significant Russian papers by A.M. Il'in and A.S. Bakhvalov analyzed new numerical methods for convection-diffusion ODEs. Bakhvalov considered an upwind difference scheme on a layer-adapted graded mesh. Such meshes are based on a logarithmic scale. In 1990 the Russian mathematician Grisha Shishkin showed that instead one could use a simpler piecewise uniform mesh. This idea has been propagated throughout the 1990s by a group of Irish mathematicians: Miller, O'Riordan, Hegarty and Farrell. Late 20th-century mathematicians who have worked on numerical methods for convection-diffusion problems include Goering, Tobiska, Roos, Lube, Felgenhauer, John, Matthies, Risch and Schieweck [10]. With the advance of unprecedented computing power, there has been a flow of literatures on numerical solutions from the nineteen eighties. Miller, O'Riordan and Shishkin constructed the Shishkin-type mesh to gain the independence of error estimation with respect to the singular perturbation parameter. Schultz and his students [11], successfully developed the stabilized high order finite difference methods. Lin, Schultz and Zhang developed boundary layer detection theory from improved a priori bounds for quasilinear singular perturbation problems. Zhang, Schultz and Lin developed sharp a priori bounds for semi-linear singular perturbation problems including ones with multiple boundary layers. [7], [8], [9]

V. A FIRST ORDER FINITE DIFFERENCE METHOD

The finite difference techniques are based upon the approximations that allow to replace the differential equations by finite difference equations. These finite difference approximations are in algebraic form, and the unknowns solutions are related to grid points. Thus, the finite difference solution basically involves three steps:

1. We define the sequence of the meshes on the solution domain [a, b]=[0,1].
2. We approximate the given differential equation by the system of difference equations that relates the solutions to grid points.
3. We solve the above algebraic system of the equations.

Let

$\overline{\Omega}_u^N = \left\{x_i = a + ih, i = 0, \ldots, N, h = \frac{b-a}{N}\right\}$ be an equidistant mesh, where $N \in R$, h is the step-size of mesh. If we use the forward-difference formula $x_i' = \frac{x_{i+1}-x_i}{h}$ or the backward-difference formula $x_i' = \frac{x_i - x_{i-1}}{h}$ with $x_i'' = \frac{x_{i+1}-2x_i+x_{i-1}}{h^2}$ in equations (1) we will get the first order finite difference method however $\varepsilon = 1$.[12]

VI. THE PROBLEM (Uniform meshes inappropriateness for upwind operator)

Let take for simplicity linear convection-diffusion in one dimension with Dirichlet boundary conditions:

$$L_\varepsilon u_\varepsilon = \varepsilon u_\varepsilon'' + b(x) u_\varepsilon' = f(x), x \in \Omega \qquad (2a)$$

$$u_\varepsilon(0) = A, u_\varepsilon(1) = B, \qquad (2b)$$

Where

$b, f \in C^2(\Omega)$, $b(x) \geq \beta > 0, x \in \overline{\Omega}$. Where the $\overline{\Omega}_u^N$ is the uniform mesh on the interval [a, b]=[0,1] impose a uniform mesh $x_i = a + ih,\ i = 0, 1, \ldots, n+1,$

The parameter h is called the mesh-size, and the points x_i are the mesh points. $\overline{\Omega}_u^N = \left\{x_i = \frac{i}{N}, i = 0, \ldots, N\right\}$

The backward-difference formula is the following:

Theorem: Let L_ε be the differential operator in (2a) and $v \in \overline{\Omega}$. If $v(0) \geq 0, v(1) \geq 0\ and\ L_\varepsilon v(x) \leq 0\ for\ all\ x \in \Omega, then\ v(x) \geq 0, for\ all\ x \in \overline{\Omega}$.

Proof: found out in p.14 of the reference [6]

We see that the exact solution $u_\varepsilon(x)$ of (2a-b) for $x \neq 0$, in general $v_0(0) \neq u_\varepsilon(0), for\ \varepsilon > 0$ and so boundary layer occurs at the boundary point $x = 0$, and u_ε and its derivatives, for all integers $k \geq 0$, satisfy the bounds

$$\left|u_\varepsilon^{(k)}(x)\right| \leq C\left(1 + \varepsilon^{-k} e^{-\beta x/\varepsilon}\right), for\ all\ x \in \overline{\Omega}. \qquad (3)$$

Where C is constant independent of ε. It follows that, outside the small open neighborhood $(0, k\beta^{-1}\varepsilon \ln(1/\varepsilon))$ of the boundary point $x = 0$, in other words outside the boundary layer, the solution and its derivatives are ε-uniformly bounded in the sense that, for all k,

$$\sup_{x \geq \beta^{-1}\varepsilon \ln(1/\varepsilon)} \left|u_\varepsilon^{(k)}(x)\right| \leq C, \qquad (4)$$

Where C is independent of ε, and

$$\varepsilon^{-k} e^{-\beta x/\varepsilon} \leq \varepsilon^{-k} e^{-\ln(1/\varepsilon)^k} = 1 \qquad (5)$$

On other hand, for $x \leq \varepsilon$, the derivative grow without as the parameter ε tends to zero. At $x = 0$ it self we see that

$$\left|u_\varepsilon^{(k)}(0+)\right| \leq C\varepsilon^{-k}$$

Upwind finite difference operator on uniform mesh for problem (2):

$$L_\varepsilon^N U_\epsilon = \varepsilon \delta^2 U_\epsilon + b(x) D^+ U_\epsilon = f(x_i), \qquad x_i \in \Omega_u^N, \qquad (6a)$$

$$U_\varepsilon(0) = u_\varepsilon(0), U_\varepsilon(1) = u_\varepsilon(1), \qquad (6b)$$

Where the $\overline{\Omega}_u^N$ is the uniform mesh

$$\overline{\Omega}_u^N = \left\{x_i = \frac{i}{N}, i = 0, \ldots, N\right\} \qquad (6c)$$

The non-zero entries of the system matrix A^N associated with this finite difference method are

$$a_{i,i-1} = \frac{\varepsilon}{h^2}, a_{i,i} = \frac{\varepsilon}{h^2}(2 + \rho_i),\ a_{i,i+1} = \frac{\varepsilon}{h^2}(1 + \rho_i),$$

Where $\rho_i = a x_i h/\varepsilon$, here A^N is irreducibly diagonally dominant and also that, irrespective of the value of ε, the sign pattern of typical row of this tridiagonal matrix is (+,-,+). This shows that $-A^N$ is an M-matrix, and so the finite difference operator L_ε^N in (6a) satisfies a discrete minimum principle. Also the discrete minimum principle established directly without system matrix through proving the theorem below by contradiction.

Definition: The finite difference method with the associated system matrix A^N is monotone if either A^N or - A^N is a monotone matrix.

Definition: The finite difference $L^N V = A^N V = f$ satisfies discrete minimum principle**,** if, for any mesh function *V*, the inequalities

$min(V(x_0), V(x_N)) \geq 0$ and $L^N V(x_i) \leq 0$ for all $x_i \in \Omega^N$, imply that

$$V(x_i) \geq 0 \text{ for all } x_i \in \overline{\Omega}^N,$$

Theorem: The upwind finite difference operator L_ε^N in (3a) satisfies the discrete minimum principle.

Proof: found out in the p.23 of reference [6].

If we use the method (2) to solve the specific problem

$$\varepsilon u_\varepsilon'' + 2u_\varepsilon' = 0, x \in \Omega, u_\varepsilon(0) = 0, u_\varepsilon(1) = 0 \qquad (7a)$$

with the exact solution:

$$u_\varepsilon(x) = \frac{e^{-2x/\varepsilon} - e^{-2/\varepsilon}}{1 - e^{-2/\varepsilon}} \qquad (7b)$$

Then

$$U_\varepsilon(x_i) = \frac{\lambda^i - \lambda^N}{1 - \lambda^N}, \text{where } \lambda = \frac{1}{1 + 2/\varepsilon N}.$$

It is clear that $0 < \lambda < 1$, and that $U_\varepsilon(x_i)$ is monotone decreasing with increasing x_i, which implies that no oscillations occur. However, despite of absence of numerical oscillations, this method is not satisfactory in the sense that it is not an ε-uniform method. To see this we consider the error at the first mesh point $x_1 = 1/N$. This is given by

$$U_\varepsilon(x_1) - u_\varepsilon(x_1) = \frac{\lambda - \lambda^N}{1 - \lambda^N} - \frac{r - r^N}{1 - r^N}, where\ r = e^{-2/\varepsilon N}$$

Taking $\varepsilon N = 1$ and letting $N \to \infty$, we see that this error tends to non-zero quantity $1/3 - e^{-2} = 0.197998$, which proves that method (2) is not an ε-uniform for problem (3) and that the maximum pointwise error is about 20% no matter how large N is.[6]

VII. CONVERGENCE OF UNIFORM MESH BACKWARD FINITE DIFFERENCE METHOD

With regard to norm which is used in these types of problems like equation (1), the reference [6], resolved by the favor of the use of any Norm, doesn't involve averaging namely maximum norm, which is defined by

$$\|u\|_{\infty} = \max_{x\in[0,1]} |u(x)|. \qquad (8)$$

In this norm we see that differences between distinct functions are detected, irrespective of how small ε is, which means that the maximum norm is an appropriate norm for the study of boundary layer phenomena. The forward or backward difference formula has an error of the order *O(h).* But if a finite difference technique applied for singularly perturbed problems like (1) the order of convergence differ to the normal cases of linear boundary value problem without the parameter ε. [13 INT]

A finite difference method is a discretization of the differential equation using the grid points x_i, where the unknowns u_i *(for i=0, ..., N)* are approximations of the values $u(x_i)$.

Classical convergence theory for finite difference methods is based on complementary concepts of consistency and stability first formally we write (1) as:

$$L_h u_h = f_h, \qquad (9)$$

Where L_h is a matrix,

$$u_h = (u_h(x_0), u_h(x_1), \dots, u_h(x_N))^T = (u_0, u_1, \dots, u_N)^T,$$

And $f_h = (f(x_0), f(x_1), \dots, f(x_N))^T$. Functions defined on the grid, such as u_h, f_h, are called grid functions. The restriction of a function $v \in (0,1)$ to a grid function is denoted by $R_h v$, $viz\ R_h v = (v(x_0), v(x_1), \dots, v(x_N))$. We some time omit R_h when the meaning is clear. The discrete maximum norm on the space of grid functions is

$$\|v_h\|_{\infty,d} = \max_i |v_h(x_i)| \qquad (10)$$

Under the assumption $u \in C^4[0,1]$, the central difference scheme (9) is consistent of order two, also a discrete problem $L_h u_h = f_h$, is stable in the discrete maximum norm, if there exist a constant K (the stability constant) that is independent of h, such that

$$\|u_h\|_{\infty,d} \le K \|L_h u_h\|_{\infty,d} \qquad (11)$$

for all mesh functions u_h. The main result of classical convergence theory for finite difference method is:

Consistency+stability ⇒ convergence. For all sufficiently small ***h*,** the backward difference scheme for the boundary value problem (1) is stable in the discrete maximum norm. One can clearly combine consistency and stability to obtain a first-order convergence result. [6], [3]

VIII. CONVERGENCE OF PIECEWICE UNIFORM MESH BACKWARD FINITE DIFFERENCE METHOD

Consider a family of mathematical problems parameterized by a singular perturbation parameter ε, where ε lies in the semi-open interval 0< ε≤1. Assume that each problem in the family has a unique solution denoted by u_ε, and that each u_ε is approximated by a sequence of numerical solutions $\left\{U_\varepsilon, \overline{\Omega}^N\right\}_{N=1}^{\infty}$ where U_ε is defined on the mesh $\overline{\Omega}^N$ and N is a discretization parameter. Then, the numerical solutions U_ε are said to converge ε-uniformly to the exact solution u_ε, if there exist a positive integer N_0, and positive numbers C and p, where N_0, C and p are all independent of N and ε, such that, for all $N \ge N_0$,

$$\sup_{0<\varepsilon\le 1} \|U_\varepsilon - u_\varepsilon\|_{\overline{\Omega}_\tau^N} \le CN^{-p} \qquad (12)$$

or

$$\sup_{0<\varepsilon\le 1} \|U_\varepsilon - u_\varepsilon\|_{\overline{\Omega}_\tau^N} \le Ch^{p} \qquad (13)$$

Here p is called the *ε-uniform rate or order of convergence* and *C* is called the *ε-uniform error constant*. Classical finite difference methods cannot be expected to be ε-uniform on a uniform mesh if a standard upwind finite difference method is applied to the linear convection-diffusion problem (P_ε) [6]

We recall a result about the behavior with respect to ε of the exact solution of problems class (1) and its derivatives, which we will use in the convergence analysis. In [14], [15] it was shown that, with smooth enough data, the solution of (2) can be written as $u = v + w$, where the regular component v and the singular component w satisfy $L_\varepsilon v = f, L_\varepsilon w = 0$, respectively, with appropriate boundary conditions such that for $0 \le k \le l$ (l is an integer depending on the regularity) it holds equation(2)

$$|v^k(x)| \le C, |w^k(x)| \le C\varepsilon^{-k} e^{-2\alpha(1-x)/\varepsilon} \qquad (14)$$

Consider a family of singularly perturbed boundary value problems ODE (1) denoted by P_ε depending on a small parameter ε. Under many conditions, a solution $u_\varepsilon(x)$ of P_ε can be constructed by the well-known method of perturbation i.e., as a power series in ε with first term $u_0(x)$ being the solution of the problem as ε=0 i.e.,

$$-bu' + cu = f \;\; in\ (0,1), \;\; u(0) = \gamma 0,$$
$$u(1) = \gamma 1 \qquad (15)$$

The main unifying features of problems having two or more limit process expansions is that certain terms in the governing differential equation will change their orders of magnitude depending on the domain in *x*. Often, the highest derivative in the differential equation will be multiplied by the small parameter *ε*, and this term will be small everywhere except near special points, e.g., boundary points. The difficulty near *x*= 0 arises from the fact that the differential equation with *ε*→0 is first order, so that the initial conditions, cannot both be satisfied. The loss of an initial or boundary condition in a problem leads, in general, to the occurrence of a boundary layer [16]. Hence when such an expansion converges as ε→0 uniformly in *x*, we have a regular perturbation problem. When $u_\varepsilon(x)$ does not have a uniform limit in *x* as ε→0, i.e. [17]

$$\lim_{x\to 0}\left(\lim_{\varepsilon\to 0} u_\varepsilon(x)\right) = u_0(0) \ne \lim_{\varepsilon\to 0}\left(\lim_{x\to 0} u_\varepsilon(x)\right) = \alpha_0 \quad (16)$$

For error analysis in [18], [19] and [20 INT] a priori parameter explicit bounds on the solution of singularly perturbed elliptic problems of convection–diffusion type are established and parameter-uniform numerical methods for a singularly perturbed elliptic problem with parabolic boundary layers in the solution are analyzed. We have inserted the Shishkin refined strategy blow into a Matlab program of backward finite difference with basically uniform mesh, namely in the book [1] based on a class of Boundary Value Methods as in equation (1) and on a linearization strategy.

IX. ASSIGNING THE TYPE OF THE LAYER

Moreover, we shall assume that in [0,1], c(x) and $f(x)$ are continuous and, for simplicity, $b(x)$ is differentiable. The behavior of the solution depends, of course, on the properties of the functions $b(x)$ and $c(x)$. There exist subintervals inside [0, 1] where the solution vary rapidly (layers), they may be localized either at the extreme points of the interval [0,1] (boundary layers) or near the roots x_i of $b(x)$, which are called turning points (interior layers), and the special case of this last (centered layer).

X. NOTE

1- If $b(x) = 0, \forall x \in [0,1]$ here the problem in equation (1) called reaction diffusion, but if $b(x) \neq 0 \text{ or } \exists x \in [0,1] \; s.t \; b(x) = 0$, the problem (1) called convection-diffusion and the zeros called turning points of the problem.

2- We can verify the existences, types and locations of subintervals of [0, 1] where the solution vary rapidly (layers), essentially taken from $b(x), \acute{b}(x)$ and $c(x)$ signs, in class of problems as in equation (1) as in follows table:

Table(1): Existences, types and locations of layers about problems as terms of equation (1).

$b(x) \neq 0;$ $0 \leq x \leq 1$	$b(x) < 0$, boundary layer at $x = 0$ $b(x) > 0$, boundary layer at $x = 1$
$b(x) = 0;$	$c(x) > 0$, boundary layer at $x = 0$ and $x = 1$ $c(x) < 0$, rapidly oscillatory solution $c(x)$ changes sign, (turning points)
$\acute{b}(x_i) \neq 0,$ $b(x_i) = 0$	$\acute{b}(x_i) > 0$ no boundary layers, interior layer at x_i $\acute{b}(x_i) < 0$ possible boundary layers, no interior layer at x_i

3- The transformation $x \to 1 - x$ reduces the case $b < 0$ to $b > 0$. [6],[21],[22]

XI. DESCRIPTION OF THE METHOD

We now describe the Shishkin mesh for convection-diffusion problem as in equations (1). Let $q \in (0,1) \text{ and } \sigma > 0$ be two mesh parameters. We define a mesh transition point λ by

$$\lambda = min\left\{q, \frac{\sigma\varepsilon}{\beta} ln\, N\right\} \qquad (17)$$

where

$$\beta = \underset{0 \leq x \leq 1}{\text{minbnd}}\, b(x)$$

Then the intervals $[0, \lambda]$ and $[\lambda, 1]$ are divided into qN and $(1-q)N$ equidistant subintervals (assuming that qN is an integer). This mesh may be regarded as generated by the mesh generating function

$$\varphi(\xi) = \begin{cases} \frac{\sigma\varepsilon}{\beta}\tilde{\varphi}(\xi) \text{ with } \tilde{\varphi}(\xi) = \ln N \frac{\xi}{q} \text{ for } \xi \in [0,q], \\ 1 - \left(1 - \frac{\sigma\varepsilon}{\beta} ln\, N\right)\frac{1-\xi}{1-q} \quad \text{for} \quad \xi \in [q,1] \end{cases} \qquad (18)$$

if q ≥ λ; Again the parameter q is the amount of mesh points used to resolve the layer. The mesh transition point λ has been chosen such that the layer term $\exp(-\beta x/\varepsilon)$ is smaller than $N^{-\alpha}$ on $[\lambda, 1]$. Typically σ will be chosen equal to the formal order of the method or sufficiently large to accommodate the error analysis. [23 INT] The course part of this Shishkin mesh has spacing h=(1-q)(1- λ)/N, so $N^{-1} \leq h \leq qN^{-1}$. The fine part has spacing $h = q\lambda/N = q(\frac{\sigma}{\beta})\varepsilon N^{-1} \ln N$, so $h \ll \varepsilon$. Thus there is a very abrupt change in mesh size as one passes from the coarse part to the fine part. The mesh is not locally quasi-equidistant, uniformly in ε. On the mesh $x_i = ih \text{ for } i = 0, \dots, N/2$ and $x_i = 1 - (N - i)h$ for $i = \frac{N}{2} + 1, \dots, N$.

A key property, nonequidistant of the Shishkin mesh, for convection diffusion-problems are some time described as "layer resolving" meshes. One might infer from this terminology that wherever the derivatives of *u* are large, the mesh is chosen so fine that the truncation error of the difference scheme is controlled. But the Shishkin mesh does not fully resolve the layer: for

$$|u'(x)| \approx C\varepsilon^{-1}e^{-b(1)(1-x)/\varepsilon}$$

so

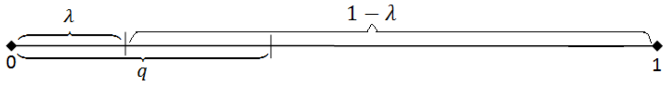
$$|u'(1-\lambda)| \approx C\varepsilon^{-1}e^{-2lnN} = C\varepsilon^{-1}N^{-2}$$

Which in general is large since typically $\varepsilon \ll N^{-1}$ that is $|u'(x)|$ is still large on part of the first coarse-mesh interval $[x_{N/2-1}, x_{N/2}]$.[21]

XII. CONSTRUCTION THE PIECEWICE UNIFORM SHISHKIN FITTED MESH [24], [3], [25]

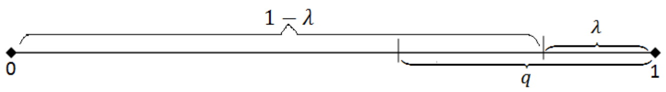
We will use four construction of Shishkin piecewise uniform fitted mesh ,as decoding of the Shishkin function in equation (18) , each of, which is a mesh vary depending on the location of the singularity, as follows: 19a, 19b, 19c and 19d , below represents mapping to fix the location of boundary and interior layers, puts fine part of the mesh, of thickness not exceeding the value of transition point indicator λ as in equation(17), at the left, the right, the center, and both extreme points (left and right) respectively.

$$x_i = \frac{\lambda}{qN} i, x_{j+qN} = 1 - \lambda + \frac{(1-\lambda)}{(1-q)N} j, i = 0,1,\dots,qN, j = 0,1,\dots,(1-q)N \quad (19a)$$



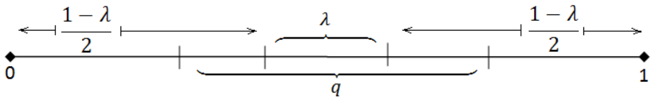
$$x_i = \frac{(1-\lambda)}{(1-q)N} i, \qquad x_{j+(1-q)N} = 1 - \lambda + \frac{\lambda}{qN} j \qquad (19b)$$

$$i = 0, 1, \dots, (1-q)N, \qquad j = 0, 1, \dots, qN$$



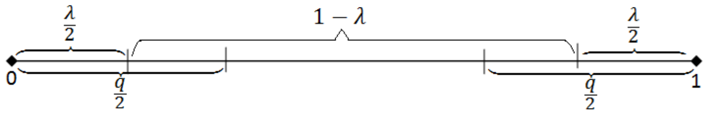
$$x_i = \frac{(1-\lambda)}{(1-q)N} i, x_{j+\frac{(1-q)}{2}N} = \frac{1-\lambda}{2} + \frac{\lambda}{qN} j, x_{i+\frac{1+q}{2}N} = \frac{1+\lambda}{2} + \frac{\lambda}{qN} i \qquad (19c)$$

$$i = 0, 1, \dots, \frac{(1-q)}{2}N, \; j = 0, 1, \dots, qN$$



$$x_i = \frac{\lambda}{qN} i, x_{j+\frac{q}{2}N} = \frac{\lambda}{2} + \frac{1-\lambda}{(1-q)N} j, x_{i+(1-\frac{q}{2})N} = 1 - \frac{\lambda}{2} + \frac{\lambda}{qN} i \qquad (19d)$$

$$i = 0, 1, \dots, \frac{q}{2}N, \qquad j = 0, 1, \dots, (1-q)N$$



XIII. THE OUTLINE

To find the approximation solution to the problem of equation (2) with Shishkin fitted Mesh:

1- Decide on how many mesh points (sub intervals multiplicand number N),
2- Determine the Shishkin transition point indicator λ as in equation (17).
3- Allocate fine part of Shishkin mesh on the interval [0,1], corresponding to singularly boundary layer dropping on X-axis, which is easy process by one of these options:
   a- Going on the non equality (16) at both extreme points.
   b- Going on $b(x), \acute{b}(x)$ and $c(x)$ signs as in the equations as in table(1).
   c- Otherwise we can deducing it through applying, uniform mesh with the backward finite difference method numerical solution, once and observing the plot of the exit solution with the boundary conditions to discover lineament of location of the singular (stiff) layers.
4- The first improvement is by little tuning the value of σ in equation (17) until the solution became softer. The second improvement let $0 < v < 1$ be arbitrary real number and *$[f_1,f_2]$ be* the fine subinterval with length $D = f_2 - f_1$:

a- For equations 19a, 19b and 19d we divide the fine subinterval to two another non equal subintervals as follows:

$$[f_1, f_2] = [f_1, f_1 + vD] \cup [f_1 + vD, f2]$$

Then the fine subinterval itself is divided into fine and coarse regions depending on the value of v in the arrangement and intensity without a change in the total length D since $D = vD + (1-v)D$, in the case of if $v = 1$ then the mesh will return to normal construction of the Shishkin mesh.

b- For equations 19c we divide the fine subinterval let denote it by *$[f_1,f_2]$* to three another non equal subintervals as follows:

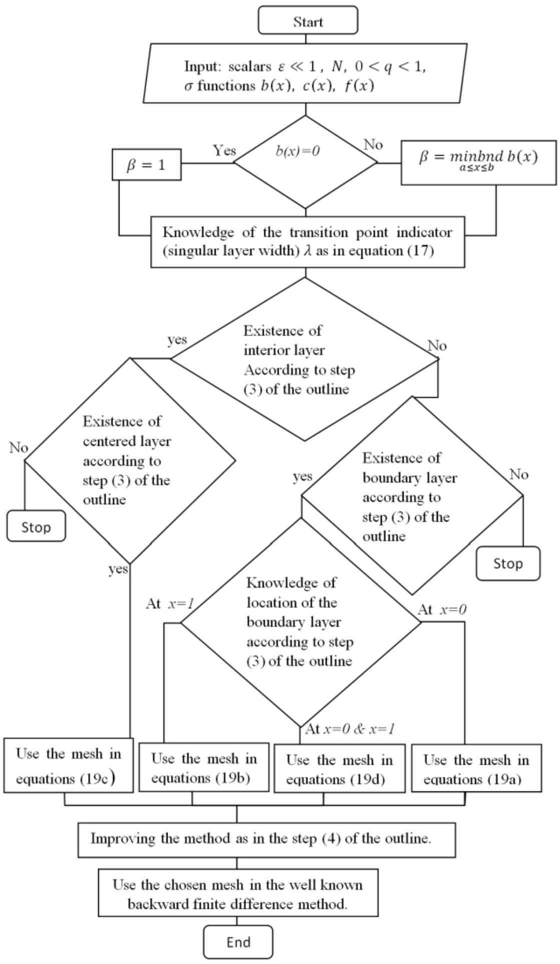
$$[f_1, f_2] = \left[f_1, f_1 + \frac{v}{2}D\right] \cup \left[f_1 + \frac{v}{2}D, f_2 - \frac{v}{2}D\right] \cup \left[f_2 - \frac{v}{2}D, f_2\right]$$

Then the fine subinterval itself is divided into three (two of them same) fine and coarse regions depending on the value of *v* in the arrangement and intensity without a change in the total length *D* since

$D = \frac{v}{2}D + (1-v)D + \frac{v}{2}D$, in the case of if $v = 1$ then the mesh will return to normal construction of the Shishkin mesh.

5- Apply the steps of **bacward finite difference method** to find the approximation solution.

XIV. FLOWCHART OF THE NEW METHOD



XV. TEST PROBLEM [26]

1) $-\varepsilon u''(x) + u'(x) - (1+\varepsilon)u(x) = 0, u(-1) = 1 + e^{-2}, u(1) = 1 + e^{-2(1+\varepsilon)/\varepsilon}$

$$u^E(x) = e^{(x-1)} + e^{-(1+\varepsilon)(1+x)/\varepsilon}$$

2) $-\varepsilon u''(x) + u'(x) = 0, u(0) = 1, u(1) = e^{-1/\varepsilon}$

$$u^E(x) = e^{-\frac{x}{\varepsilon}}$$

3) $-\varepsilon u''(x) - u'(x) = 0, u(0) = 1, u(1) = 0,$

$$u^E(x) = (1 - e^{(x-1)/\varepsilon})/(1 - e^{-1/\varepsilon}).$$

4) $-\varepsilon u''(x) - u(x) = 0, u(0) = 1, u(1) = 0$

$$u^E(x) = e^{-x/\sqrt{\varepsilon}} - \frac{e^{((x-2)/\sqrt{\varepsilon})}}{1 - e^{-2/\sqrt{\varepsilon}}}.$$

5) $-\varepsilon u''(x) + u(x) = -1, u(0) = 0, u(2) = 0$

$$u^E(x) = -x + \frac{2e^{\frac{2}{\varepsilon}}}{e^{\frac{2}{\varepsilon}} - 1} - \frac{2e^{\frac{2}{\varepsilon}}}{e^{\frac{2}{\varepsilon}} - 1} e^{-\frac{x}{\varepsilon}}$$

6) $-\varepsilon u''(x) - xu'(x) - u(x) = -(1 + \varepsilon\pi^2)\cos(\pi x) - x\sin(\pi x), u(-1) = -1, u(1) = -1$

$$u^E(x) = \cos(\pi x).$$

7) $-\varepsilon u''(x) + \frac{4x}{\varepsilon + x^2} u'(x) + \frac{2}{\varepsilon + x^2} u(x) = 0, u(-1) = \frac{1}{1+\varepsilon}, u(1) = \frac{1}{1+\varepsilon}.$

$$u^E(x) = \frac{1}{\varepsilon + x^2}.$$

8) $-\varepsilon u''(x) + xu'(x) = 0, u(-1) = 0, u(1) = 2$

$$u^E(x) = 1 + \frac{x\,erf\left(\frac{x}{\sqrt{2\varepsilon}}\right)}{erf\left(\frac{1}{\sqrt{2\varepsilon}}\right)}$$

9) $-\varepsilon u''(x) - u(x) = -(1 + \varepsilon\pi^2)\cos(\pi x), u(-1) = 0, u(1) = 0$

$$u^E(x) = \cos(\pi x) + e^{\frac{(x-1)}{\sqrt{\varepsilon}}} + e^{\frac{-(x+1)}{\sqrt{\varepsilon}}}$$

XVI. NNUMERICAL RESULT

We present the computational performance of a Matlab implementation on a set of 270648 singularly perturbation BV ($test\ problem \times \varepsilon \times N \times algorithm$) as in table (2). The Matlab implementations based on the implementation of the first order (backward) finite difference method-uniform mesh provided by [27], [28] without slightest changes in composition of the backward finite difference operator except the mesh was changed three time; uniform mesh represented in equation(10b); as (algorithm1), Shishkin mesh represented in equations(19); as algorithm 2 and the new proposed improvement mesh represented in step(4) of the outline; as (algorithm 3). The comparisons of algorithms based on maximum error as in the equation (3).

Table (2): Total number of implementation

No. of Prob.	No. of Algorithms	No. of ε values	No. of N values	Total No. of Implementation

9	3	28	358	270648

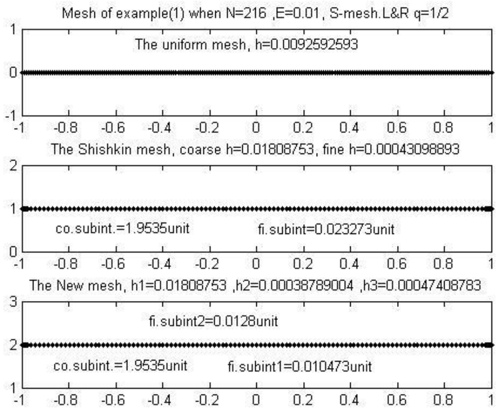
Details of table (2) are given in the following context arranged in table (3):

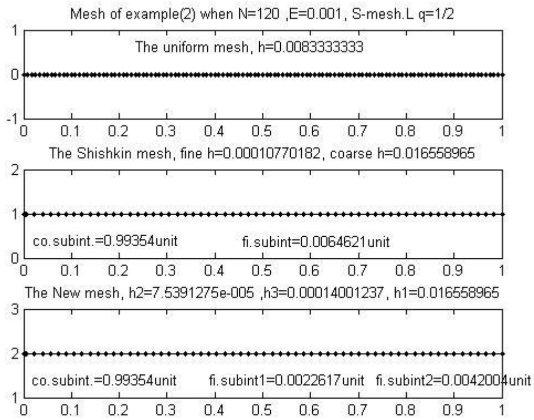
Prob. No.	Type of layer	ε	q	N							#N
1	Boundary at $x=0$ & $x=1$	1.00E-01	1/2	8	56	104	152	200			5
		1.00E-02		8	16	24	32	…	208	216	27
		1.00E-03		200	208	216	224	…	272	280	11

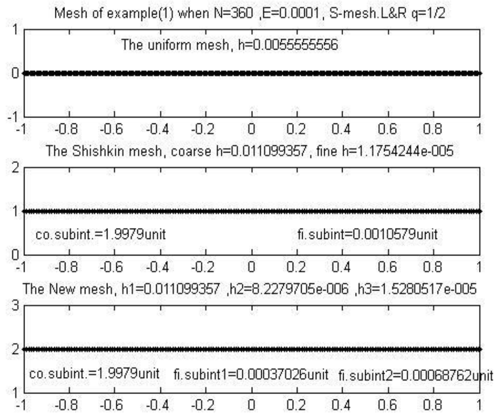
Table (3): Context of the choice of each of the fine layer location, perturbation parameter ε, Shishkin ratio q and sub intervals multiplicand number N in the numerical results and comparisons.

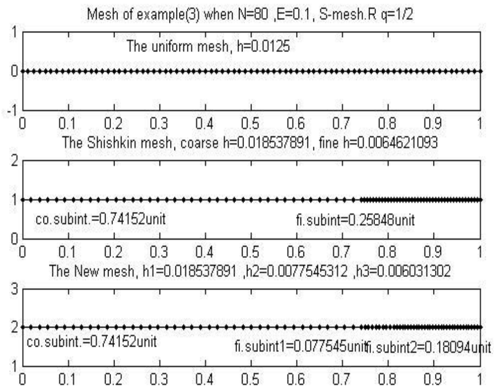
2	Boundary at $x=0$	1.00E-02	1/2	104	112	120	128	…	192	200	13
		1.00E-03		120	128	136	144	…	192	200	11
		1.00E-04		144	152	160	168	...	216	224	11
		1.00E-05		488	496	504	512	…	560	568	11
		1.00E-06		1560	1568	1576	1584	…	1632	1640	11
3	Boundary at$x=1$	1.00E-01	1/2	8	16	24	32	…	72	80	10
		4.64E-02		88	96	104	112	…	152	160	10
4	Boundary at $x=0$	1.00E-01	1/2	8	16	24	32	40	48	56	7
		1.00E-02		48	56	64	72	…	104	112	9
		1.00E-03		40	48	56	64	…	104	112	10
		1.00E-04		8	16	24	32	…	152	160	20
		1.00E-05		8	16	24	32	…	152	160	20
		1.00E-06		160	176	192	208	…	464	480	21
		1.00E-07		56	72	88	104	…	216	232	12
5	Boundary at $x=0$	1.00E-01	1/2	8	16	24	32	…	112	120	15
		1.00E-02		8	16	24	32	…	112	120	15
6	Boundary at $x=1$	1.00E-01	1/2	80	88	96	104	…	152	160	11
		1.59E-02		88	96	104	112	…	176	184	13
7	Centered	1.00E-01	1/8	280	288	296	304	…	352	360	11
		1.00E-02	1/2	1552	1560	1568	1576	1584	1592	1600	7
8	Boundary at $x=1$	1.00E-01	1/8	400	408	416	424	…	472	480	11
		1.00E-02		400	408	416	424	…	472	480	11
		1.00E-03		880	960	1040	1120	…	1520	1600	10
9	Boundary at $x=0$ & $x=1$	11.006E-05	1/2	800	808	816	824	…	864	872	10
		11.006E-06		360	368	376	384	…	544	552	25
Number of taken values of N											358

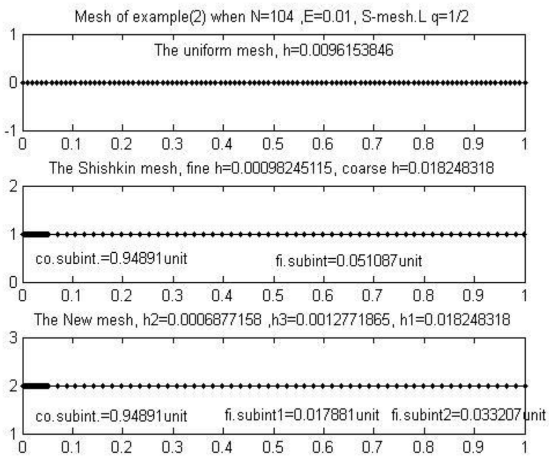
XVII. SOME MATLAB PLOTS OF MESHES CONSTRUCTION

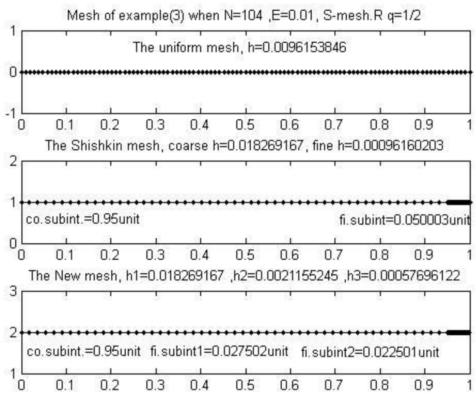

Mesh of example(1) when N=216 ,E=0.01, S-mesh.L&R q=1/2
The uniform mesh, h=0.0092592593
The Shishkin mesh, coarse h=0.01808753, fine h=0.00043098893
co.subint.=1.9535unit
fi.subint=0.023273unit
The New mesh, h1=0.01808753 ,h2=0.00038789004 ,h3=0.00047408783
fi.subint2=0.0128unit
co.subint.=1.9535unit
fi.subint1=0.010473unit

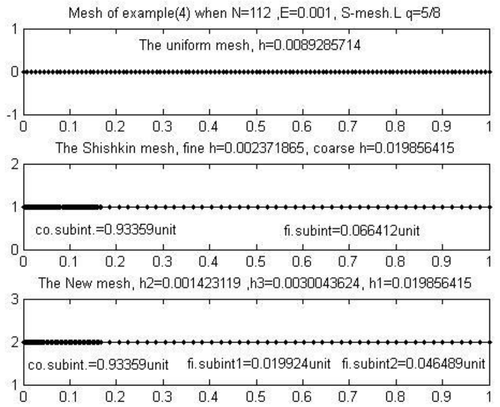

Mesh of example(2) when N=120 ,E=0.001, S-mesh.L q=1/2
The uniform mesh, h=0.0083333333
The Shishkin mesh, fine h=0.00010770182, coarse h=0.016558965
co.subint.=0.99354unit
fi.subint=0.0064621unit
The New mesh, h2=7.5391275e-005 ,h3=0.00014001237, h1=0.016558965
co.subint.=0.99354unit
fi.subint1=0.0022617unit
fi.subint2=0.0042004unit

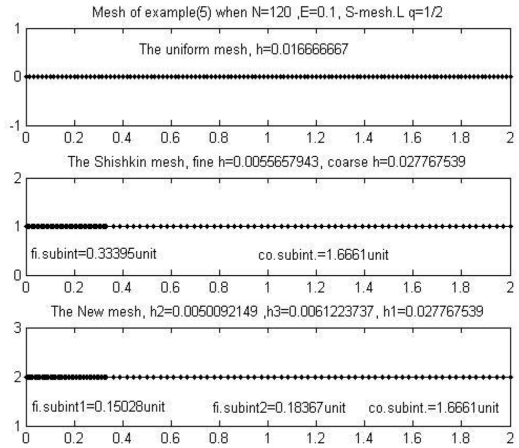

Mesh of example(1) when N=360 ,E=0.0001, S-mesh.L&R q=1/2
The uniform mesh, h=0.0055555556
The Shishkin mesh, coarse h=0.011099357, fine h=1.1754244e-005
co.subint.=1.9979unit
fi.subint=0.0010579unit
The New mesh, h1=0.011099357 ,h2=8.2279705e-006 ,h3=1.5280517e-005
co.subint.=1.9979unit
fi.subint1=0.00037026unit
fi.subint2=0.00068762unit

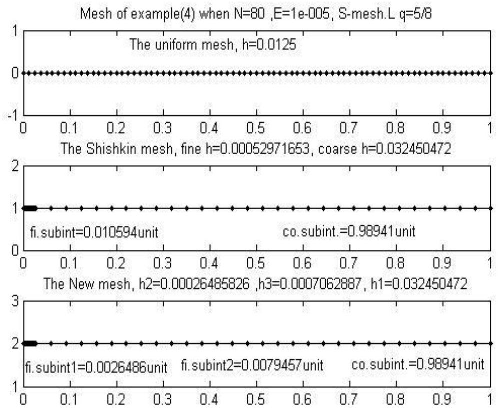

Mesh of example(3) when N=80 ,E=0.1, S-mesh.R q=1/2
The uniform mesh, h=0.0125
The Shishkin mesh, coarse h=0.018537891, fine h=0.0064621093
co.subint.=0.74152unit
fi.subint=0.25848unit
The New mesh, h1=0.018537891 ,h2=0.0077545312 ,h3=0.006031302
co.subint.=0.74152unit
fi.subint1=0.077545unit
fi.subint2=0.18094unit

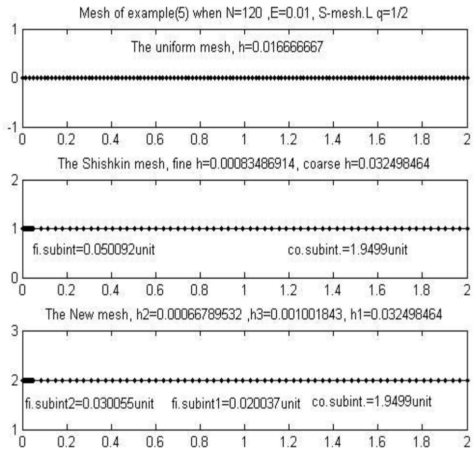

Mesh of example(2) when N=104 ,E=0.01, S-mesh.L q=1/2
The uniform mesh, h=0.0096153846
The Shishkin mesh, fine h=0.00098245115, coarse h=0.018248318
co.subint.=0.94891unit
fi.subint=0.051087unit
The New mesh, h2=0.0006877158 ,h3=0.0012771865, h1=0.018248318
co.subint.=0.94891unit
fi.subint1=0.017881unit
fi.subint2=0.033207unit

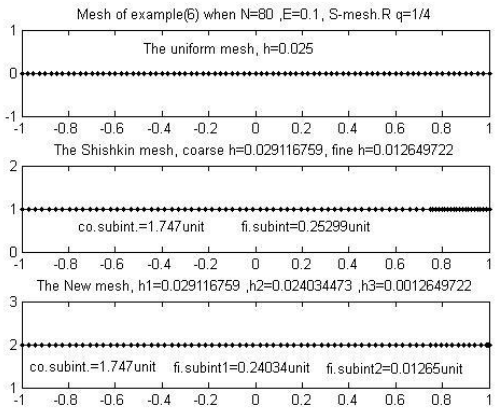

Mesh of example(3) when N=104 ,E=0.01, S-mesh.R q=1/2
The uniform mesh, h=0.0096153846
The Shishkin mesh, coarse h=0.018269167, fine h=0.00096160203
co.subint.=0.95unit
fi.subint=0.050003unit
The New mesh, h1=0.018269167 ,h2=0.0021155245 ,h3=0.00057696122
co.subint.=0.95unit
fi.subint1=0.027502unit
fi.subint2=0.022501unit

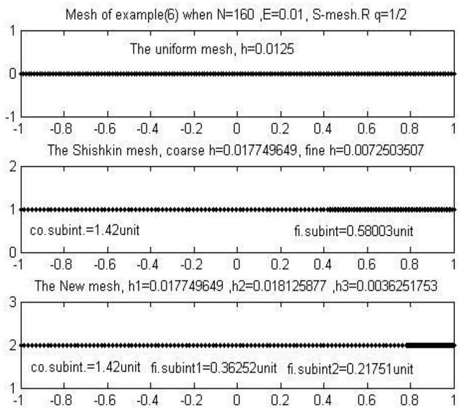

Mesh of example(4) when N=112 ,E=0.001, S-mesh.L q=5/8
The uniform mesh, h=0.0089285714
The Shishkin mesh, fine h=0.002371865, coarse h=0.019856415
co.subint.=0.93359unit
fi.subint=0.066412unit
The New mesh, h2=0.001423119 ,h3=0.0030043624, h1=0.019856415
co.subint.=0.93359unit
fi.subint1=0.019924unit
fi.subint2=0.046489unit

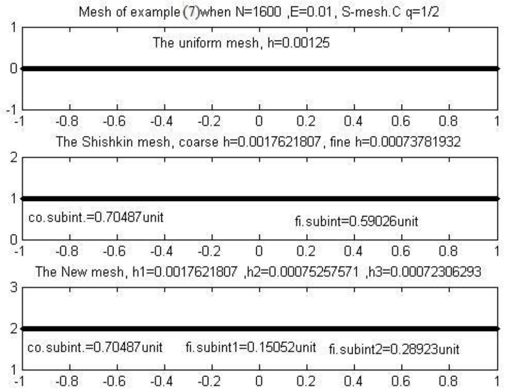

Mesh of example(5) when N=120 ,E=0.1, S-mesh.L q=1/2
The uniform mesh, h=0.016666667
The Shishkin mesh, fine h=0.0055657943, coarse h=0.027767539
fi.subint=0.33395unit
co.subint.=1.6661unit
The New mesh, h2=0.0050092149 ,h3=0.0061223737, h1=0.027767539
fi.subint1=0.15028unit
fi.subint2=0.18367unit
co.subint.=1.6661unit

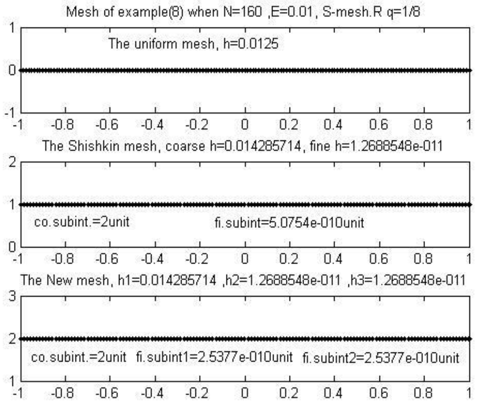

Mesh of example(4) when N=80 ,E=1e-005, S-mesh.L q=5/8
The uniform mesh, h=0.0125
The Shishkin mesh, fine h=0.00052971653, coarse h=0.032450472
fi.subint=0.010594unit
co.subint.=0.98941unit
The New mesh, h2=0.00026485826 ,h3=0.0007062887, h1=0.032450472
fi.subint1=0.0026486unit
fi.subint2=0.0079457unit
co.subint.=0.98941unit

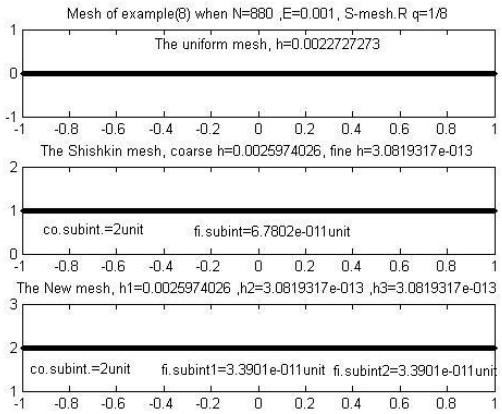

Mesh of example(5) when N=120 ,E=0.01, S-mesh.L q=1/2
The uniform mesh, h=0.016666667
The Shishkin mesh, fine h=0.00083486914, coarse h=0.032498464
fi.subint=0.050092unit
co.subint.=1.9499unit
The New mesh, h2=0.00066789532 ,h3=0.001001843, h1=0.032498464
fi.subint2=0.030055unit
fi.subint1=0.020037unit
co.subint.=1.9499unit

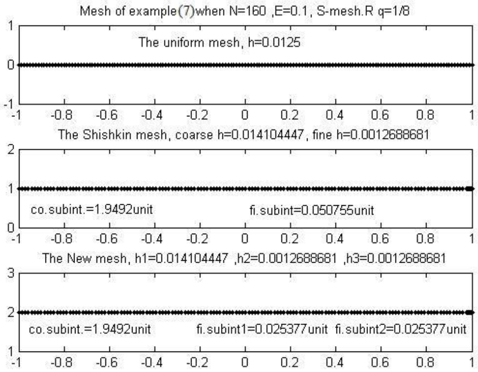

Mesh of example(6) when N=80 ,E=0.1, S-mesh.R q=1/4
The uniform mesh, h=0.025
The Shishkin mesh, coarse h=0.029116759, fine h=0.012649722
co.subint.=1.747unit
fi.subint=0.25299unit
The New mesh, h1=0.029116759 ,h2=0.024034473 ,h3=0.0012649722
co.subint.=1.747unit
fi.subint1=0.24034unit
fi.subint2=0.01265unit


Mesh of example(6) when N=160 ,E=0.01, S-mesh.R q=1/2
The uniform mesh, h=0.0125
The Shishkin mesh, coarse h=0.017749649, fine h=0.0072503507
co.subint.=1.42unit
fi.subint=0.58003unit
The New mesh, h1=0.017749649 ,h2=0.018125877 ,h3=0.0036251753
co.subint.=1.42unit
fi.subint1=0.36252unit
fi.subint2=0.21751unit

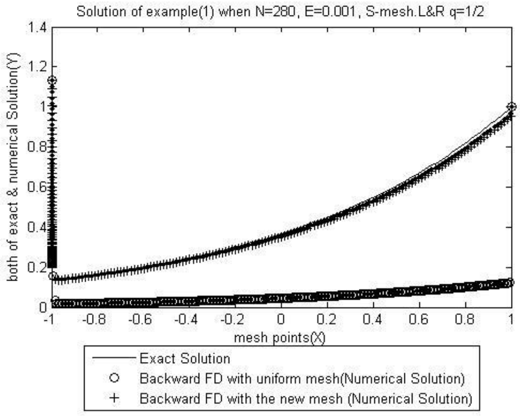


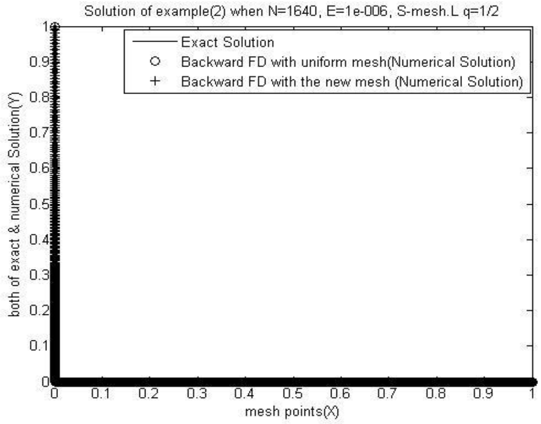


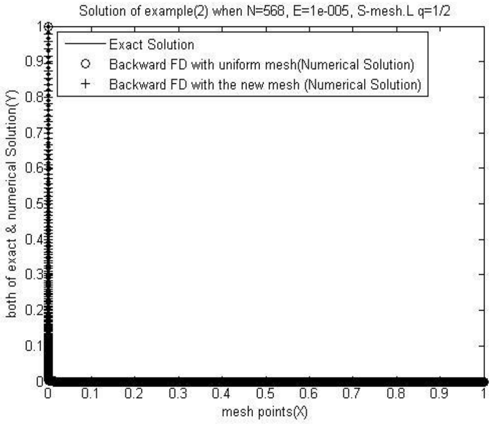


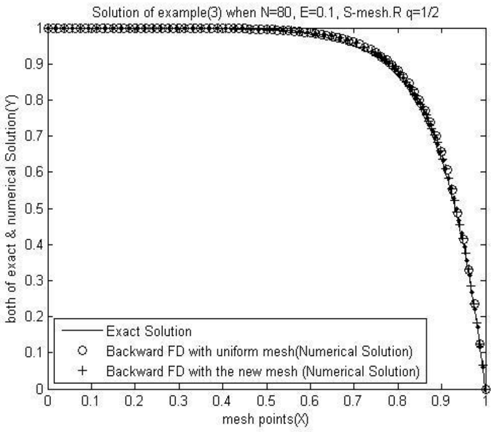


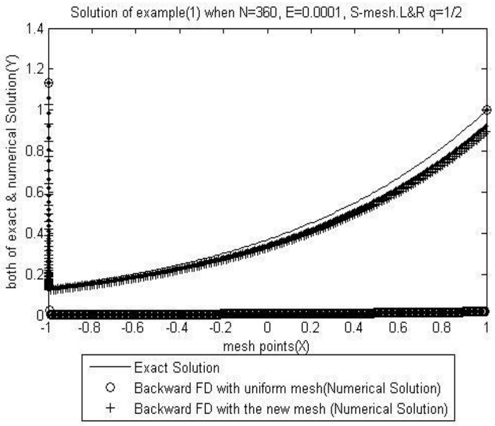
XVIII. MATLAB PLOTS ILLUSTRATE SOLUTIONS OF THE NEW ALGORITHM TOGETHER WITH THE EXACT SOLUTIONS

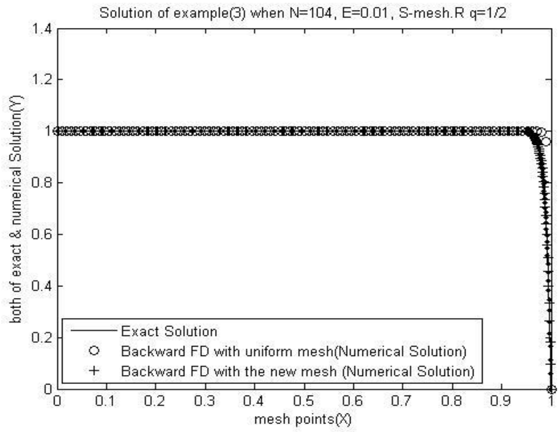

Solution of example(1) when N=280, E=0.001, S-mesh.L&R q=1/2
both of exact & numerical Solution(Y)
mesh points(X)
Exact Solution
Backward FD with uniform mesh(Numerical Solution)
Backward FD with the new mesh (Numerical Solution)

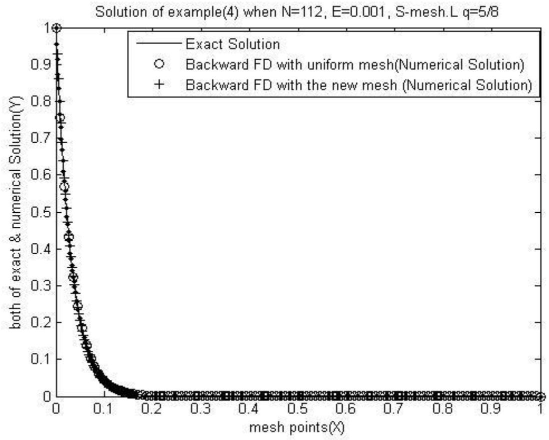

Solution of example(2) when N=1640, E=1e-006, S-mesh.L q=1/2
Exact Solution
Backward FD with uniform mesh(Numerical Solution)
Backward FD with the new mesh (Numerical Solution)
both of exact & numerical Solution(Y)
mesh points(X)

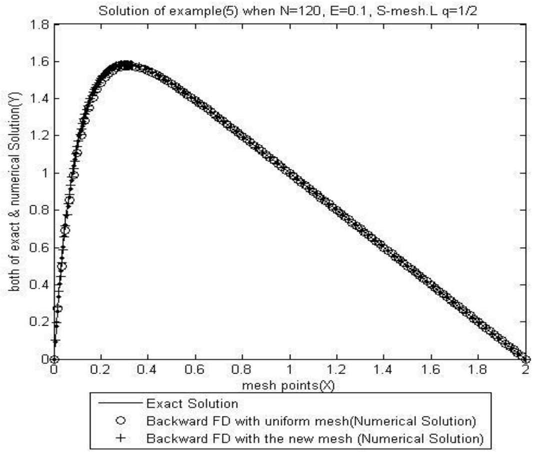

Solution of example(2) when N=568, E=1e-005, S-mesh.L q=1/2
Exact Solution
Backward FD with uniform mesh(Numerical Solution)
Backward FD with the new mesh (Numerical Solution)
both of exact & numerical Solution(Y)
mesh points(X)

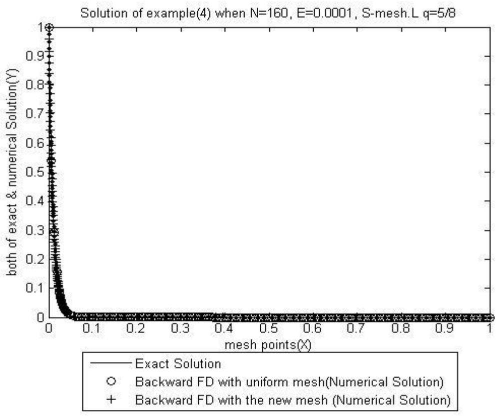

Solution of example(3) when N=80, E=0.1, S-mesh.R q=1/2
both of exact & numerical Solution(Y)
Exact Solution
Backward FD with uniform mesh(Numerical Solution)
Backward FD with the new mesh (Numerical Solution)
mesh points(X)

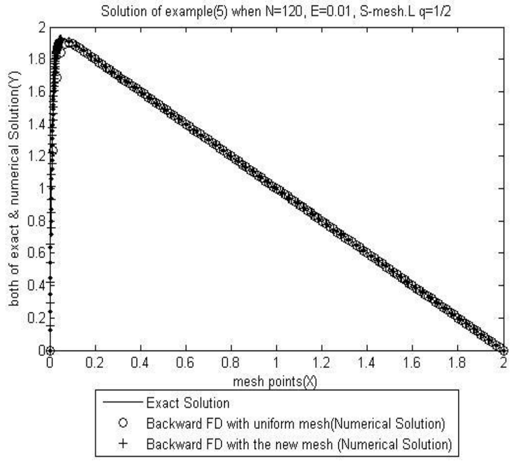

Solution of example(1) when N=360, E=0.0001, S-mesh.L&R q=1/2
both of exact & numerical Solution(Y)
mesh points(X)
Exact Solution
Backward FD with uniform mesh(Numerical Solution)
Backward FD with the new mesh (Numerical Solution)

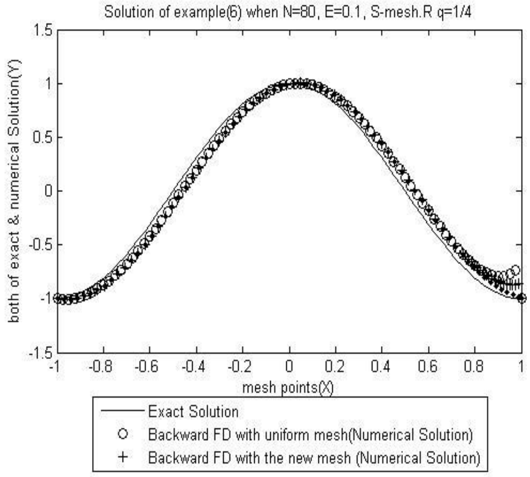

Solution of example(3) when N=104, E=0.01, S-mesh.R q=1/2
both of exact & numerical Solution(Y)
Exact Solution
Backward FD with uniform mesh(Numerical Solution)
Backward FD with the new mesh (Numerical Solution)
mesh points(X)

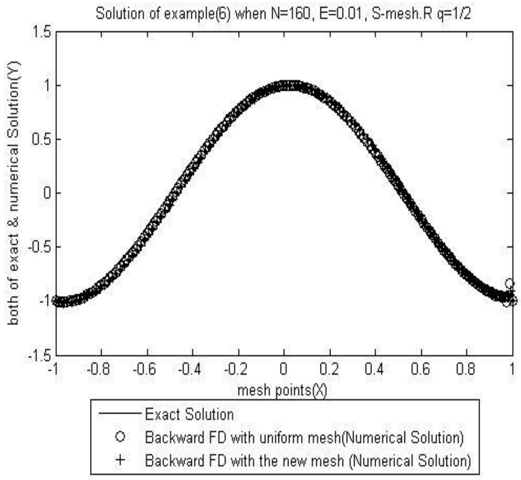

Solution of example(4) when N=112, E=0.001, S-mesh.L q=5/8
both of exact & numerical Solution(Y)
mesh points(X)
Exact Solution
Backward FD with uniform mesh(Numerical Solution)
Backward FD with the new mesh (Numerical Solution)

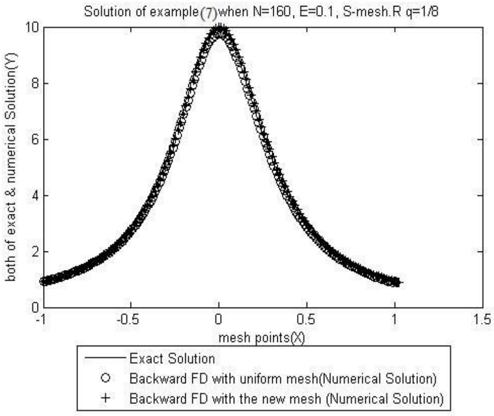

Solution of example(5) when N=120, E=0.1, S-mesh.L q=1/2
both of exact & numerical Solution(Y)
mesh points(X)
Exact Solution
Backward FD with uniform mesh(Numerical Solution)
Backward FD with the new mesh (Numerical Solution)

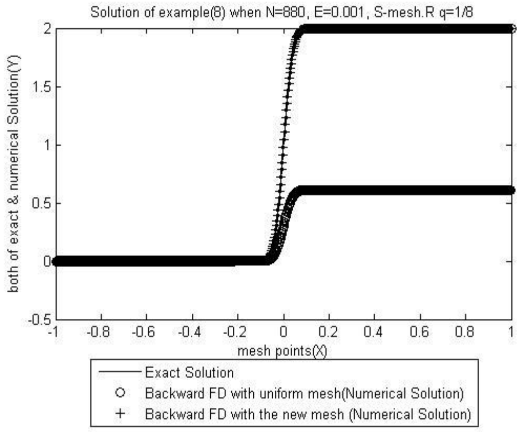

Solution of example(4) when N=160, E=0.0001, S-mesh.L q=5/8
both of exact & numerical Solution(Y)
mesh points(X)
Exact Solution
Backward FD with uniform mesh(Numerical Solution)
Backward FD with the new mesh (Numerical Solution)

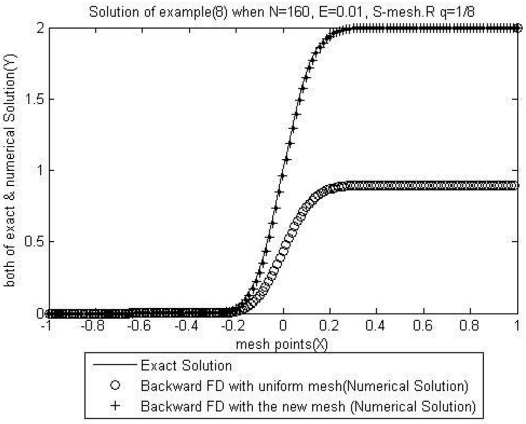

Solution of example(5) when N=120, E=0.01, S-mesh.L q=1/2
both of exact & numerical Solution(Y)
mesh points(X)
Exact Solution
Backward FD with uniform mesh(Numerical Solution)
Backward FD with the new mesh (Numerical Solution)

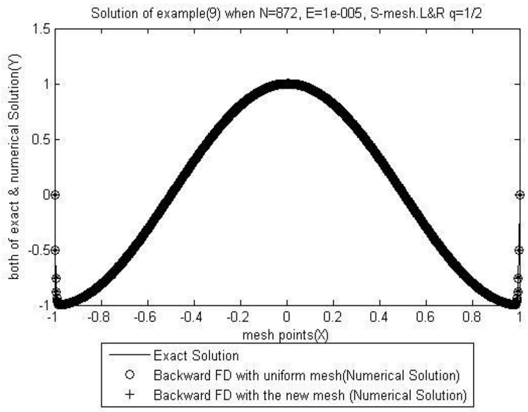

Solution of example(6) when N=80, E=0.1, S-mesh.R q=1/4
both of exact & numerical Solution(Y)
mesh points(X)
Exact Solution
Backward FD with uniform mesh(Numerical Solution)
Backward FD with the new mesh (Numerical Solution)

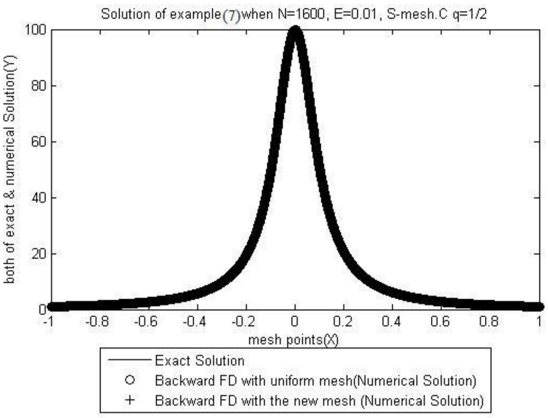

Solution of example(6) when N=160, E=0.01, S-mesh.R q=1/2
both of exact & numerical Solution(Y)
mesh points(X)
Exact Solution
Backward FD with uniform mesh(Numerical Solution)
Backward FD with the new mesh (Numerical Solution)

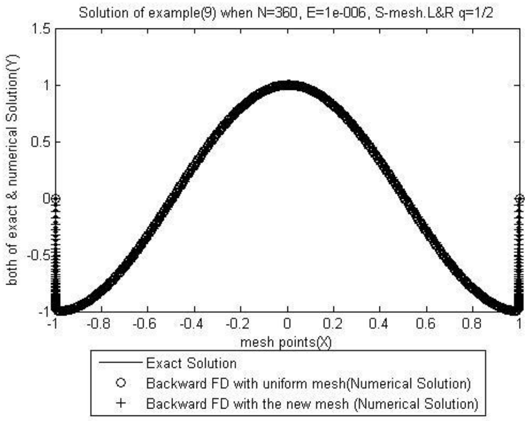




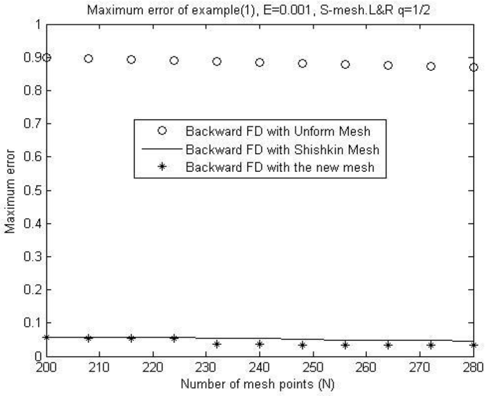


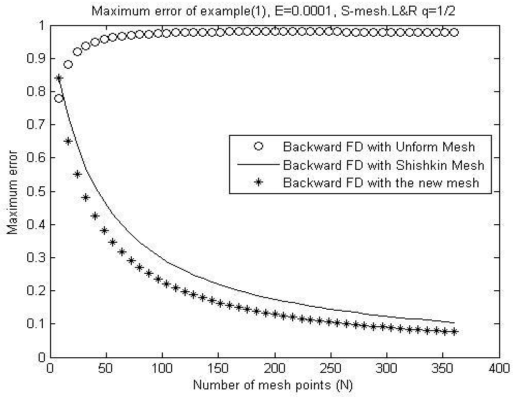


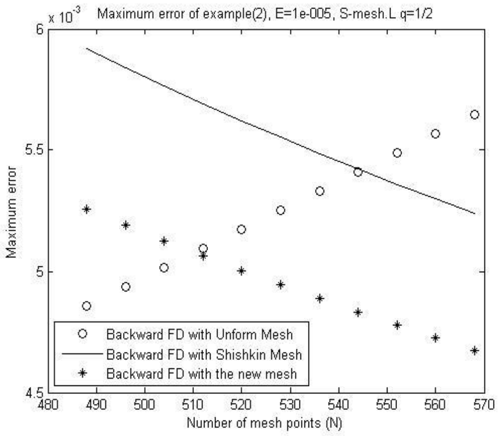


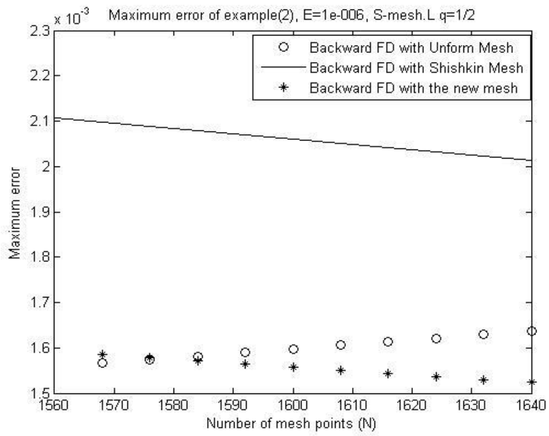


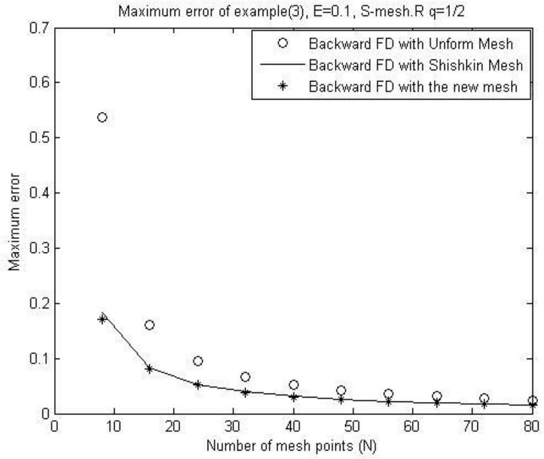
XIX. MAMTLAB PLOTS COMPARE THE CONVERGENCE OF THE THREE ALGORITHMS

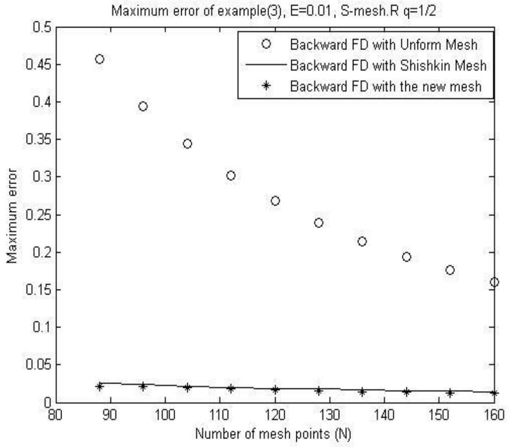

Maximum error of example(1), E=0.001, S-mesh.L&R q=1/2
Backward FD with Uniform Mesh
Backward FD with Shishkin Mesh
Backward FD with the new mesh
Maximum error
Number of mesh points (N)

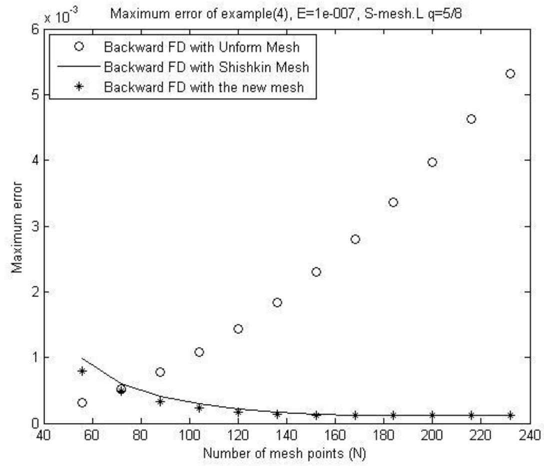

Maximum error of example(1), E=0.0001, S-mesh.L&R q=1/2
Backward FD with Uniform Mesh
Backward FD with Shishkin Mesh
Backward FD with the new mesh
Maximum error
Number of mesh points (N)

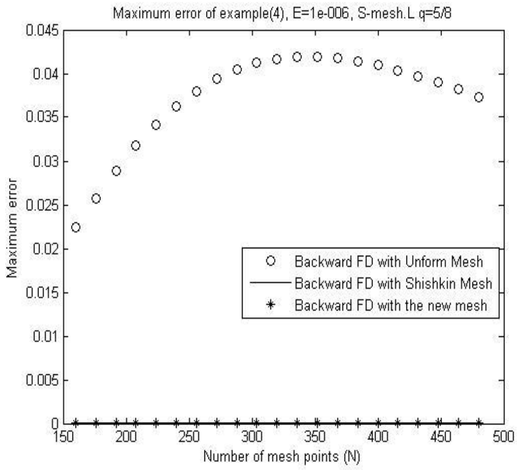

x 10^-3
Maximum error of example(2), E=1e-005, S-mesh.L q=1/2
Backward FD with Uniform Mesh
Backward FD with Shishkin Mesh
Backward FD with the new mesh
Maximum error
Number of mesh points (N)

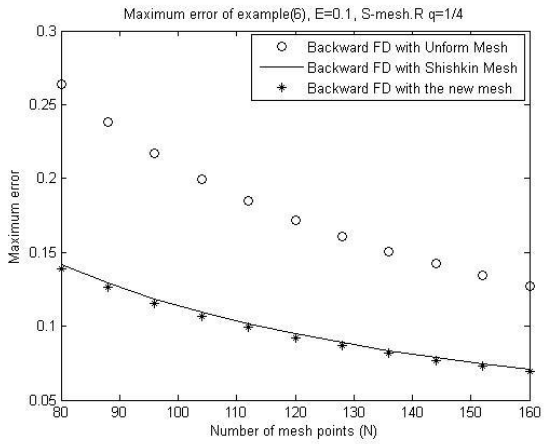

x 10^-3
Maximum error of example(2), E=1e-006, S-mesh.L q=1/2
Backward FD with Uniform Mesh
Backward FD with Shishkin Mesh
Backward FD with the new mesh
Maximum error
Number of mesh points (N)

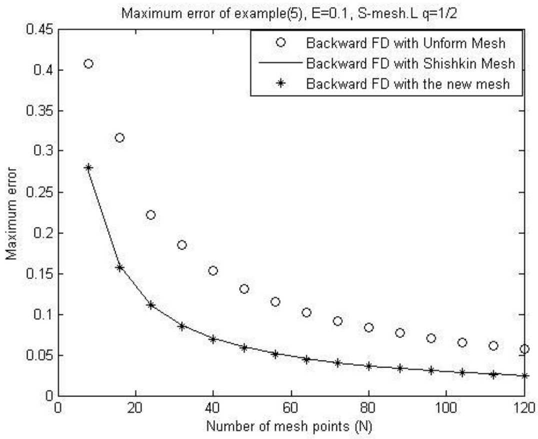

Maximum error of example(3), E=0.1, S-mesh.R q=1/2
Backward FD with Uniform Mesh
Backward FD with Shishkin Mesh
Backward FD with the new mesh
Maximum error
Number of mesh points (N)

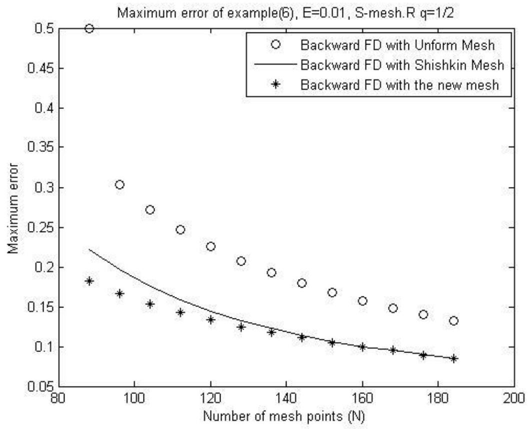

Maximum error of example(3), E=0.01, S-mesh.R q=1/2
Backward FD with Uniform Mesh
Backward FD with Shishkin Mesh
Backward FD with the new mesh
Maximum error
Number of mesh points (N)

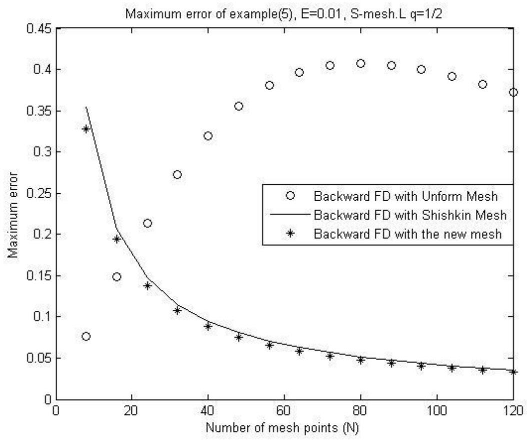

x 10^-3
Maximum error of example(4), E=1e-007, S-mesh.L q=5/8
Backward FD with Uniform Mesh
Backward FD with Shishkin Mesh
Backward FD with the new mesh
Maximum error
Number of mesh points (N)

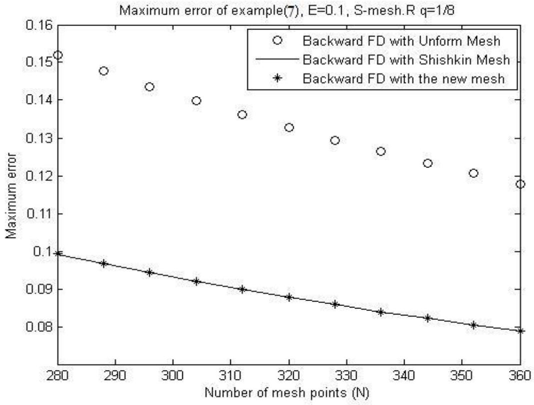

Maximum error of example(4), E=1e-006, S-mesh.L q=5/8
Maximum error
Number of mesh points (N)
Backward FD with Unform Mesh
Backward FD with Shishkin Mesh
Backward FD with the new mesh

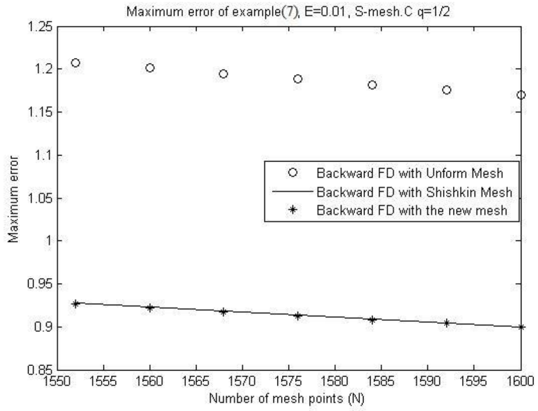

Maximum error of example(6), E=0.1, S-mesh.R q=1/4
Maximum error
Number of mesh points (N)
Backward FD with Unform Mesh
Backward FD with Shishkin Mesh
Backward FD with the new mesh

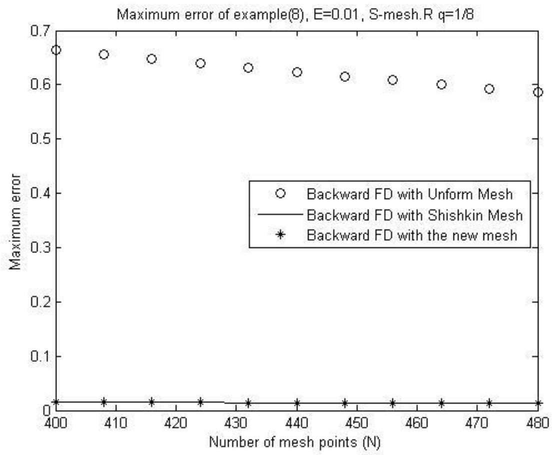

Maximum error of example(5), E=0.1, S-mesh.L q=1/2
Maximum error
Number of mesh points (N)
Backward FD with Unform Mesh
Backward FD with Shishkin Mesh
Backward FD with the new mesh

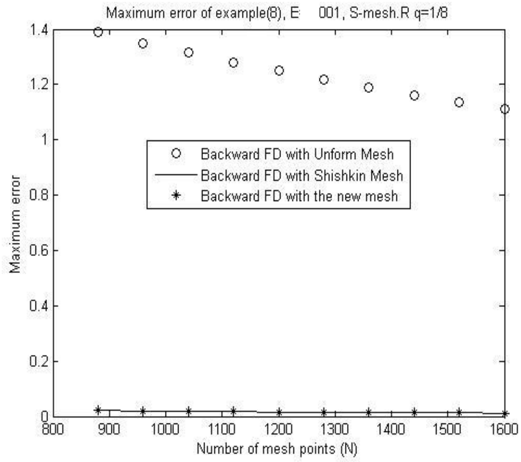

Maximum error of example(6), E=0.01, S-mesh.R q=1/2
Maximum error
Number of mesh points (N)
Backward FD with Unform Mesh
Backward FD with Shishkin Mesh
Backward FD with the new mesh

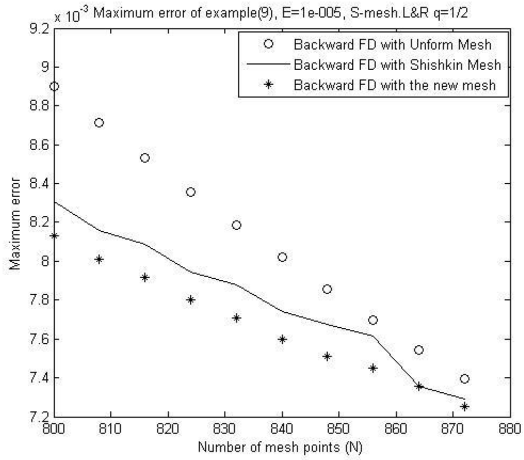

Maximum error of example(5), E=0.01, S-mesh.L q=1/2
Maximum error
Number of mesh points (N)
Backward FD with Unform Mesh
Backward FD with Shishkin Mesh
Backward FD with the new mesh

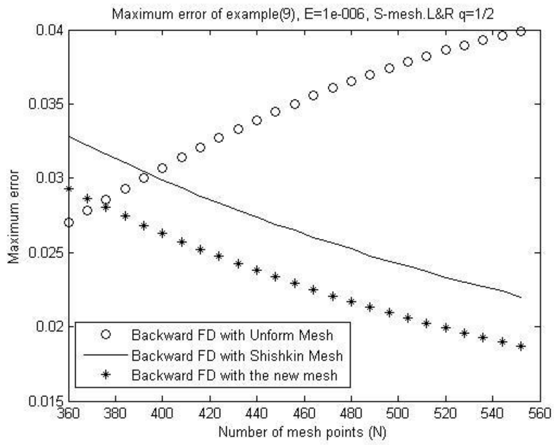












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