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Fibonacci Numbers and Golden Ratio in Mathematics and Science

Omotehinwa T. O
Department of Mathematical Sciences
Kogi State University
Anyigba, Nigeria
e-mail: oluomotehinwa {at} divasofttechnology.com

Ramon S.O
Department of Mathematics
University of Wolvehampton
West Midlands, United Kingdom

Abstract—**In this paper we discussed the mathematical concept of consecutive Fibonacci numbers or sequence which has leads to golden ratio (an irrational number that most often occurred when taking distances proportion in simple geometry shape), and convergence of the sequence and any geometric significance. We consider the spiral and self-similar curve which have been appreciated for long due to their beauty in the world. Furthermore, we take into account the recurrence relation of Fibonacci sequence, which can be use to generate Lucas sequence with the aid of different initial condition. The explanations show the ways and how Fibonacci numbers have been in existence in the world we live. And some works of art and architecture that asserts the presence of golden ratio.**

Keywords- Fibonacci numbers; Golden ratio; Lucas sequence; Geometry; Spiral and self-similar curve component

I. Introduction

Let's begin with a brief history of the outstanding and the most celebrated mathematician of the European Middle Ages, Leonardo Fibonacci. He was born around 1170 into the family of Bonacci in Pisa, which is now known as Italy. He died around 1250. Some people also called him Leonardo of Pisa. Fibonacci, a young man in his twenties acquired the early education in Bougie, where he was introduced to Hindu-Arabic numeration system and Hindu-Arabic computational technique by a Muslim schoolteacher [1]. He travelled to European countries (Greece, France, etc) and Northern part of Africa (Egypt). In his trip, he learned various arithmetic systems. When Fibonacci returned to Pisa, he was convinced and has superiority of the Hindu-Arabic notation numbers over the Roman numeration system [2]. Around 1202, Leonard of Pisa published is first book on mathematics known as Liber Abaci which was basically on arithmetic (discourse on mathematical method in commerce) and elementary algebra. He introduced the Hindu-Arabic system of numeration and arithmetic algorithm to Europe, which is the significant part out of the two contributions he is majorly remembered for. In 1225, Fibonacci published another two books, the Flos (flowers) and the Liber Quadratorum (the book of square numbers). The two books dealt with theory of numbers. Because of Fibonacci's brilliance and originality, he outshines the abilities of the scholars of his time [1]. The other contributions which seemingly insignificant known to be brain-teaser [2] which was posted in his first book (Liber Abaci).

A man put a pair of rabbits (a male and a female) in a garden that was enclosed. How many pairs of rabbits can be produced from the original pair within 12 months, if it is assumed that every months each pair of rabbits produce another pair (a male and a female) in which they become productive in the second month and no death, no escape of the rabbits and all female rabbits must reproduced during this period (year)? [1]

The solution to this problem has a certain sequence of numbers, which today is known as Fibonacci numbers or Fibonacci sequence [3]. The (figure 1.0) shows the representation of how the rabbits reproduced.

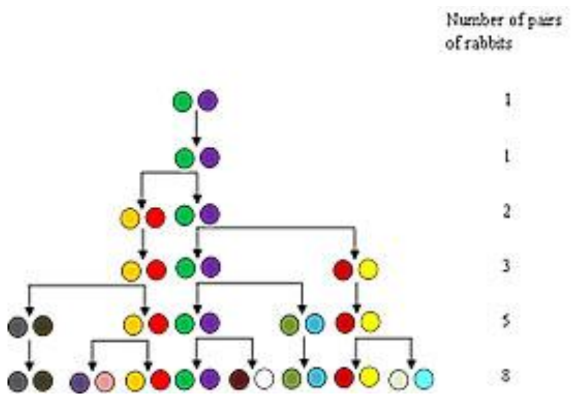

Figure 1. Shows the representation of how the rabbits are produced

Let assumed that a pair of rabbits (a male and a female) was born in January first. It will take a month before they can produce another pair of rabbits (a male and a female) which means no other pair except one in the first of February. Then, in first of March we have 2 pairs of rabbits. This will continue by having 3 pairs in April 1, 5 pairs in May 1, 8 pairs in June 1 and so on. The table below shows the total number of pairs in a year.

TABLE I. SHOWS THE TOTAL NUMBER OF PAIRS IN A YEAR

Month	Baby Rabbit	Mature Rabbit	Total
January 1	1	0	1
February 1	0	1	1
March 1	1	1	2
April 1	1	2	3
May 1	2	3	5
June 1	3	5	8
July 1	5	8	13
August 1	8	13	21
September 1	13	21	34
October 1	21	34	55
November 1	34	55	89
December 1	55	89	144

From TABLE I. above, the last column give 1, 1, 2, 3, 5, 8, 13, 21, 34 . . . what we know as Fibonacci numbers, which was named by a French mathematician Francois-Edouard- Anotole-Lucas around 1876. And we have 144 pairs of rabbits in a year. Fibonacci numbers has been one of the most interesting number sequences that will be ever written down. The Fibonacci series is unique in such a way that for every two odd numbers the next is an even number. Also, the modern scientists and physicists commonly apply the recursive series of Fibonacci sequence [4]. This sequence has gone beyond a simple arithmetic in the branches of mathematics due to the fact that it was surprisingly rediscovered in variety of forms. Fibonacci numbers has a fascinating and unique property in the sense that, for all Fibonacci numbers is the sum of the two immediately preceding Fibonacci numbers except the first two numbers [1]. Base on its methodological development has led to a great application in mathematics and computer science [2]. The importance of the sequence gave course for an organization of mathematicians to form Fibonacci Association that used to publish scholarly journal, Fibonacci Quarterly [1] and [2]

Golden Ratio

The irrational number, golden ratio [5] which is also known as golden section by the ancient Greeks [1], golden proportion, mean ratio, divine proportion, golden mean or golden number [6], is widely used in modern sciences particularly in theoretical physics [7]. The golden ratio φ, has many properties in which people are eager to know. It is a number that is equal to the reciprocal of its own with the addition of 1: $\varphi = \frac{1}{\varphi} + 1$ [8]. Likewise, the ratio of any two consecutive Fibonacci numbers converges to give approximates of 1.618, or its inverse, 0.618. This shows the relationship between Fibonacci numbers and golden ratio. This Golden ratio had been used by Egyptians in the construction of their great pyramids. It is denoted by Greek letter called phi (Φ, capital letter or φ, small letter) but φ will be used in this context. The letter φ was given name phi by a mathematician from America, Mark Barr because it is the first letter in Greek alphabets in the name of Phidias, the greatest sculptor of Greek [1], that widely used golden section principles in all his sculptures [9]. Golden ratio has unique mathematical principles in all his sculptures [9]. Golden ratio has unique mathematical properties [6]. One of the properties is the concept that was originated in plane geometry, division of a line segment into two segments.

If we can divide a line in such a way that the ratio of the whole length to the length of the longer segment happen to be equal to the ratio of the length of the longer segment to the length of the shorter segment, then we can say the ratio is golden ratio [6].

Fig. 2 below shows the division of a whole length 1 into two segments x *and* $x - 1$.

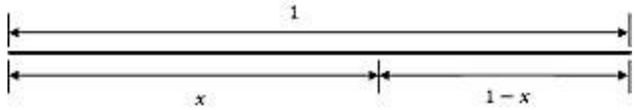

Figure 2.

This gives mean ratio if $\frac{1}{x} = \frac{x}{1-x}$ which satisfies the quadratic equation $x^2 + x - 1 = 0$ Thus, $x = \frac{-1 \pm \sqrt{5}}{2}$ But we are interested in the positive root, since lengths cannot be negative. So, $x = \frac{-1+\sqrt{5}}{2} = 0.61803398875\ldots$

Hence, the desire ratio is $\varphi = \frac{1}{x} = 1.61803\ldots$

II. FIBONACCI NUMBERS AND NATURE

The Fibonacci numbers had occurred in nature through the pattern of the petals on flowers, the arrangement of leaves in the stem of a plant, the pineapples scales or the pinecones bracts. Fibonacci sequence can also be figure out in the part human body (hands). Different answers will be given if we ask ourselves "why does the arrangement occur". For the arrangement of leaves, it might be for the space maximization for each leaf or for the average amount of sunlight falling on each leaf.

A. Fibonacci Numbers in Plants

There is occurrence of Fibonacci numbers in the growth of branches and leaves of some plants. Looking at the (figures 3-5) which show the kind of design that provide the best physical accommodation for the number of branches and leaves while sunlight exposure is being maximize [10].

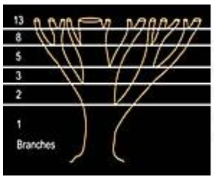

Figure 3

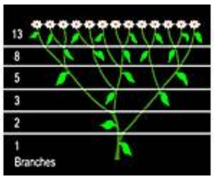

Figure 4

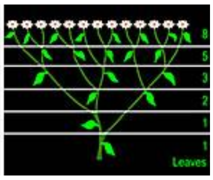

Figure 5

Source: http:// britton.disted.camosun.bc.ca/fibslide/jbfibslide.htm

The leaves on the stem of a plant get optimum exposure to sunlight, when they grow or develop in alternating or zigzags manner which form a kind of spiral pattern.

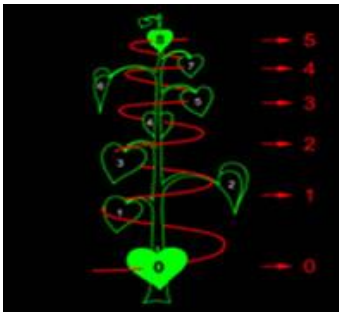

Figure 6.

Source: http:// britton.disted.camosun.bc.ca/fibslide/jbfibslide.htm

From Fig. 6, firstly, let our attention be on the leaf at the bottom of the stem in which there is a single leaf at any point and the leaf to be zero (0). Counting the leaves upward the stem until we reach the one in which is directly above the starting one, zero. The number we get, a term of the Fibonacci numbers is eight (8). And if we count the number of times we revolve around the stem, also a term of the Fibonacci numbers is five (5) [11].

B. Fibonacci Numbers in Pinecones, Sunflowers and Pineapples

Let‘s now look at the qualities of Fibonacci numbers in the case of scale pattern of tapered pinecones [10], the pattern of seed of sunflowers and the bumps on pineapples [11].
The pinecones have double set of spiral in which one of it going in clockwise direction, has 8 spirals (see Fig. 7) while the other one going in anticlockwise direction with 13 spirals (see Fig. 8). However, we can see that the Fibonacci numbers in pinecones are found to be adjacent when the spirals are counted [10].

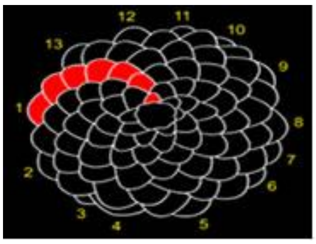

Figure 7

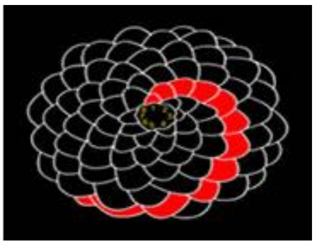

Figure 8

Source: http:// britton.disted.camosun.bc.ca/fibslide/jbfibslide.htm

Sunflowers also have two-dimensional form [11], in which there is seed arrangement of golden spiral [10]. Counting in clockwise direction we have 34 spirals and 21 spirals in the opposite directions (see Fig. 9)

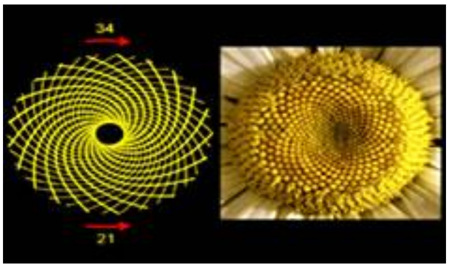

Figure 9.

Source: http:// britton.disted.camosun.bc.ca/fibslide/jbfibslide.htm

The scales on the pineapples are hexagonal in shape, with three sets of spirals [11]. The diagrams below show how the three sets being arranged.

Figure 10

Figure 11

Figure 12

Source: http:// britton.disted.camosun.bc.ca/fibslide/jbfibslide.htm

C. Fibonacci Numbers in Flowers

If we examine some flowers carefully, we will find out that the petals on a flower always have one of the Fibonacci sequence. Looking at Fig. 13, we see different flowers having petal(s) in which is a term in Fibonacci sequence.

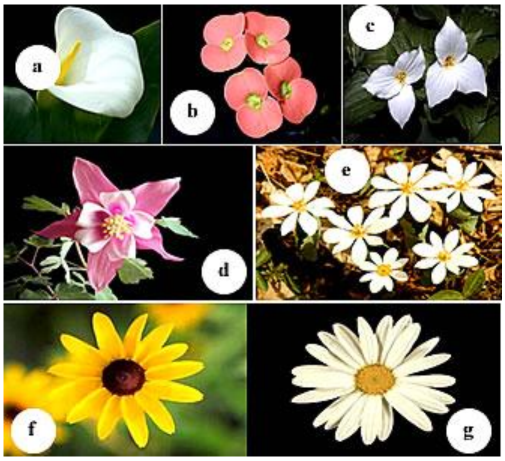

Figure 13. (a) White calla lily (1 petal) (b) Euphorbia (2 petals) (c) Trillium (3 petals) (d) columbine (5 petals) (e) Bloodroot (8 petals) (f) Black-eye susan (13 petals) (g) Shasta daisy (21 petals)

Source: http:// britton.disted.camosun.bc.ca/fibslide/jbfibslide.htm

There are very common daisies with 34 petals and also some daisies have 55 or 89 petals.

Human cannot give certain or particular answers to why nature found the arrangement of plant structures in spiral forms or shapes which is exhibiting Fibonacci numbers.

D. Fibonacci Numbers in Bees

The intriguing about Fibonacci numbers is the ways it occurs in the growing pattern of some natural dynamical system [2]. Let's examine simple model of reproductive of bees. If male bees hatch from unfertilised eggs without a father but with a mother and the female bees hatch developed from fertilised eggs having both parents (father and mother). Fig. 14 shows the family history of the bees. The unique about this family tree is that both the male and female in each generation form Fibonacci numbers and their total also gives Fibonacci series. There are more female bees than male bees in each generation except the first three. The table below shows the total number of bees up to tenth generation.

We let M and F to represent a male bee and a female bee respectively.

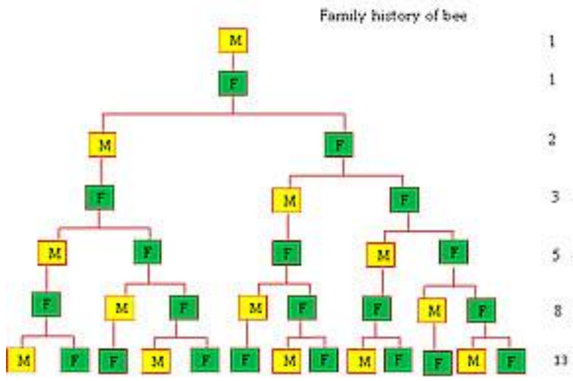

Figure 14.

E. Fibonacci Numbers in Human Hand

The uniqueness of Fibonacci numbers is that it also occurs in human hand. If we examine our hands very well we would see that each finger is separated into 3 parts with the exception of the thumbs, which is separated into 2 parts. There are 5 fingers on each hand and total of 8 fingers is separated into 3 parts. All these numbers are members of Fibonacci series [11].

III. FIBONACCI NUMBERS AND GEOMETRY

A. Fibonacci Rectangle

This is another way of occurrence of Fibonacci numbers on plane geometry. The Fibonacci rectangle can be defined as the set of rectangles that the sides are two successive Fibonacci sequence in length and which generated through squares with sides that are Fibonacci numbers. For instance, if we place two squares as size 1 (1 unit) next to each other, and since $1 + 1 = 2$ then we can draw another square (2 units) on top of the first two squares of size 1. This can be continuing by drawing a square of 3 units to touch a unit square and latter square of side 2. The 5 units square can be form by drawing a square touching both the 2 units square and 3 units square. By adding more squares all around the picture, so that the new square will be having one side which is as long as the sum of the latest two square's sides.

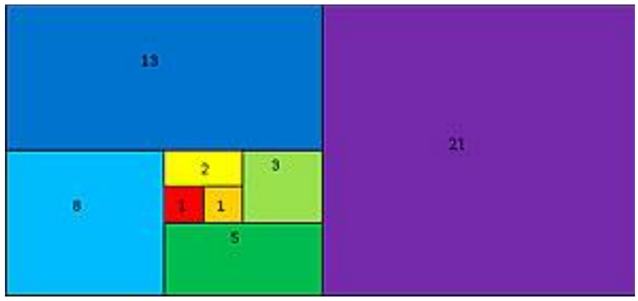

Figure 15.

B. Fibonacci Spiral

The spiral is a form of pattern in nature which is known as self-similarity [2] that grows or develops in size without changing of shape or by maintaining the same shape. For example, as current move to the ocean and tides rode unto the shore, the waves that bring in the tide curved into the spirals that can be mathematically diagrammed at a point 1,1, 2, 3, 5, 8, 13, 21, 34 and 55. Spiral can be created by drawing arcs connecting the opposite corners of squares in the Fibonacci tilling with Fibonacci numbers.

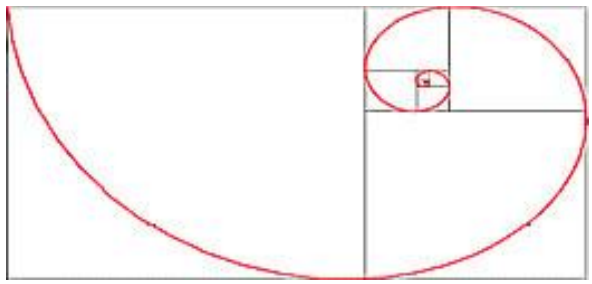

Figure 16.

Source: *http://mathworld.wolfram.com/GoldenRatio.html*

There are some organisms that grow in spiral form. For example, the shell of mollusks, "the internal surface of mollusk is smooth, the outside one is fluted. The mollusk body is inside shell and the internal surface of shells should be smooth. The outside edges of the shell augment a rigidity of shells and, thus, increase its strength. The shell forms astonish by their perfection and profitability of means spent on its creation. The spiral's idea in the expressed in the perfect geometrical form, in surprising beautiful, sharpened designed. The shells of most mollusks grow in the logarithmic spirals. These animals grow with simple mathematical calculation. The spiral shape can also be seen on the shell of snails and the sea shell. Fig. 17 pictures nautiluses which grow in spiral form.

Figure 17. Nautiluses

Source: *http://mathworld.wolfram.com/GoldenRatio.html*

The spirals shape is pleasing, because one can visually follow them with ease. The spirals base on the golden ratio contains the most incomparable structures one can find in nature. The examples are the spirals on the sequences on sunflowers, pinecones and so on.

C. Fibonacci Numbers in Pascal Triangle

A great French mathematician, Blaise Pascal who was interested in mathematics and theology, developed a triangular array known as Pascal's triangle. In which the mysterious sequence, Fibonacci numbers pop up. These Fibonacci numbers was found by adding the diagonals numbers in Pascal's triangle (see figure 3.3). The Pascal's triangle has some properties and uses. One of the properties is the addition of any two successive numbers in the diagonal (1, 3, 6, 10, 15, 21, 28, 36 …) gives the perfect square results (1, 4, 9, 16, 25, 36, 49, 64 …). It is being used for finding the probability combination problem. The major number theorist, Fibonacci is unaware of the connectivity between his rabbit's problem and theory of probability. But this was discovered 400 years later [1].

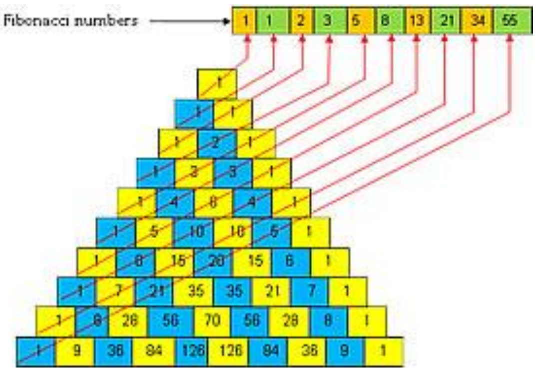

Figure 18. Pascal Triangle

D. Recursive Relation of Fibonacci Numbers

The Fibonacci numbers can be generated recursively through the initial conditions of the first two terms of Fibonacci numbers to be 1.

We let F_n represent the nth term of the Fibonacci numbers.

So, using the initial conditions;

$F_1 = 1$ $\qquad$ $F_2 = 1$

The formula for recurrence relation to generate Fibonacci sequence is

$$F_n = F_{n-1} + F_{n-2} \qquad \text{where } n \geq 3 \qquad (1)$$

We can use the recurrence relation to find the number of pairs of rabbits in a year. This is done by let n to be number of months, F_{n-1} is the number of mature pairs of rabbits in month n and F_{n-2} is the baby pairs of rabbits in n months. While the F_n is the total pairs of rabbits in months n.

Hence, in the third month; $F_3 = F_2 + F_1 = 2$

Fourth month; $F_4 = F_3 + F_2 = 3$

Fifth month; $F_5 = F_4 + F_3 = 5$

.
.
.

Twelfth month; $F_{12} = F_{11} + F_{10} = 144$

E. Lucas Numbers

The Lucas numbers can also be constructed with the aid of recursive definition of Fibonacci numbers. Where nth term of Lucas numbers is L_n, with $L_1 = 1$ and $L_2 = 3$ as the initial conditions.

Then, $L_n = L_{n-1} + L_{n-2} \qquad \text{where } n \geq 2 \qquad (2)$

Thus, the results from (2) for the Lucas sequence are 1, 3, 4, 7, 11, 17, 28 and so on. Lucas numbers is related to Fibonacci numbers in the sense that- each number in the Lucas sequence is the sum of two successive preceding digits.

IV. FIBONACCI NUMBERS AND GOLDEN RATIO

A. Relationship of Fibonacci Numbers and Golden Ratio

If we ask ourselves "what do the pyramid in Egypt, portrait of Mona Lisa, sunflowers, snails, pinecones and fingers all have in common?" The answer to this question is hidden in the series of numbers known as Fibonacci numbers. This Fibonacci numbers has interesting features; each number in the sequence equal to the sum of the two previous one;

1
1
1 + 1 = 2
1 + 2 = 3
2 + 3 = 5
3 + 5 = 8
5 + 8 = 13
8 + 13 = 21, and so on.

These numbers go to infinity which is also called Fibonacci series. Another intriguing about Fibonacci numbers is when you divide one number in the series by the number before it and as the sequence increase, $\frac{F_{n+1}}{F_n}$; you will obtain a number very close to one and another (see TABLE II) i.e. appears that $\frac{F_{n+1}}{F_n}$ approaches a limit which begins with 1.618 and it is known as golden ratio.

TABLE II. THE FIRST 30 OF FIBONACCI NUMBERS

$\frac{F_{n+1}}{F_n}$	$\frac{F_{n+1}}{F_n}$
$\frac{1}{1}$ = 1.000000000000000	$\frac{1597}{987}$ = 1.618034447821682
$\frac{2}{1}$ = 2.000000000000000	$\frac{2584}{1597}$ = 1.618033813400125
$\frac{3}{2}$ = 1.500000000000000	$\frac{4181}{2584}$ = 1.618034055727554
$\frac{5}{3}$ = 1.666666666666667	$\frac{6765}{4181}$ = 1.618033963166707
$\frac{8}{5}$ = 1.600000000000000	$\frac{10946}{6765}$ = 1.618033998521803
$\frac{13}{8}$ = 1.625000000000000	$\frac{17711}{10946}$ = 1.618033985017358
$\frac{21}{13}$ = 1.615384615384615	$\frac{28657}{17711}$ = 1.618033990175597
$\frac{34}{21}$ = 1.619047619047619	$\frac{46363}{28657}$ = 1.617859510765258
$\frac{55}{34}$ = 1.617647058823529	$\frac{75025}{46363}$ = 1.618208485214503
$\frac{89}{55}$ = 1.618181818181818	$\frac{121393}{75025}$ = 1.618033988670443
$\frac{144}{89}$ = 1.617977528089888	$\frac{196418}{121393}$ = 1.618033988780243
$\frac{233}{144}$ = 1.618055555555556	$\frac{317811}{196418}$ = 1.618033988738303
$\frac{377}{233}$ = 1.618025751072961	$\frac{514229}{317811}$ = 1.618033988754323
$\frac{610}{377}$ = 1.618037135278515	$\frac{832040}{514229}$ = 1.618033988748204

Since $\varphi = \frac{1+\sqrt{5}}{2} = 1.61803398875\ldots$ Then we can accept the prediction that the ratio converges to the limit φ, from the positive root of the quadratic equation $\varphi^2 - \varphi - 1 = 0$

This can be show by using $\varphi = \frac{F_{n+1}}{F_n}$ From the Fibonacci recursive definition, we have

$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}$$

$$\frac{F_{n+1}}{F_n} = 1 + \frac{1}{\left({F_n}/{F_{n-1}}\right)}$$

Taking $n \to \infty$, we have $\varphi = \frac{F_{n+1}}{F_n}$ which can be writing as $\varphi^2 - \varphi - 1 = 0$

Hence, $\varphi = \frac{1 \pm \sqrt{5}}{2}$ Since the desire limit is not negative.

Then, $\varphi = \frac{1+\sqrt{5}}{2} = \lim_{n \to \infty} \frac{F_{n+1}}{F_n}$ as we wanted [1].

B. Golden Ratio and Newton's Method

Around 1999, a teacher at Lasalle High School in Pennsylvania, Roche J.W. tried to show how the golden ratio can be generated by using one of the popular methods of approximation, Newton's method. During the process of estimating the golden ratio using the function $f(x) = x^2 - x - 1$, a remarkable relationship appears between the various approximations and Fibonacci sequence.

This recursive formula of Newton's method was used;

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

And $x_1 = 2$ as the seed. Then, the next three approximations will be $x_2 = \frac{5}{3}$, $x_3 = \frac{34}{21}$ and $x_4 = \frac{1597}{987}$.

Since these approximations show the consecutive ratios of Fibonacci numbers, then Roche conjectured that $x_n = \frac{F_{2^n+1}}{F_{2^n}}$ where $n \geq 1$

Roche established the validity of his conjecture by using principle mathematical induction.

Since $x_1 = 2 = \frac{F_3}{F_2} = \frac{2}{1}$, means the formula is valid for $n = 1$

Let assumed it is true for n. $x_n = \frac{F_{p+1}}{F_p}$ where p = 2^n Since $f'(x) = 2x - 1$

Hence,

$$x_{n+1} = x_n - \frac{x_n^2 - x_n - 1}{2x_n - 1} = \frac{x_n^2 + 1}{2x_n - 1} = \frac{F_{p+1}^2/F_p^2 + 1}{2F_{p+1}/F_p - 1} = \frac{F_{p+1}^2 + F_p^2}{F_{p(2F_{p+1} - F_p)}}$$

$$= \frac{F_{2p+1}}{F_{p(F_{p+1} + F_{p-1})}} = \frac{F_{2p+1}}{F_p L_n} = \frac{F_{2p+1}}{F_{2p}}$$

As a result of mathematical induction the formula is valid for every $n \geq 1$ [1].

C. Fibonacci Numbers and Prime Numbers

There are many prime numbers in Fibonacci sequence. For example, we have 2, 3, 5, 13, 89, 233, 1597, 28657 and so on which are member of Fibonacci series. It is greatly accepted that there are infinitely many Fibonacci numbers are primes in which their proofs still hard to find [1]. The largest known Fibonacci prime is F_{81839}, which was discovered by David Broadbent and Bouk de Water around 2001 [12]. Each Fibonacci number has been marked in a special way. For instance, if we examine the prime factors of a Fibonacci number, we will see that at least one of the prime factors which have never appeared before as factor in any of the earlier Fibonacci numbers. This is called Carmichael's theorem, in which it is applicable to all Fibonacci numbers [12] F_n. But with the exception of the following four special cases;

i. $F_1 = 1$ (no prime factors)
ii. $F_2 = 1$ (no prime factors)
iii. $F_6 = 8$ (this has prime factor 2 which is also the F_3)
iv. $F_{144} = 12$ (this has prime factors 2 and 3 in which they first appeared at F_3 and F_4 respectively)

The TABLE III shows prime factors of Fibonacci numbers and the bold typed are those in which have never appeared before.

TABLE III. THE FIRST 25 OF FIBONACCI NUMBERS

n	***F_n***	**Prime/Prime factor**
1	1	No prime
2	1	No prime
3	2	2
4	3	3
5	5	5
6	8	2^3
7	13	13
8	21	3 x **7**
9	34	2 x**17**
10	55	5 x **11**
11	89	89

12	144	2^4 x 3^2
13	233	233
14	377	13 x **29**
15	610	2 x 5 x **61**
16	987	3 x 7 x **47**
17	1597	1597
18	2584	2^3 x 17 x **19**
19	2181	**37** x **113**
20	6765	3 x 5 x 11 x **41**
21	10946	2 x 13 x **421**
22	17711	89 x **199**
23	28657	28657
24	46368	2^5 x 3^2 x 7 x **23**
25	75025	5^2 x **3001**

Nevertheless, as mentioned above that for every two odd numbers in Fibonacci sequence the next is an even number. The Fibonacci even is shown with colour red in the table 4.1.
Another interesting about Fibonacci numbers is that, they have been patterned in such a way that the multiples of 3 in the series have equal interval and likewise the multiples of 5. The TABLE IV only shows the multiples of 3 and multiples of 5 in the first 30 of Fibonacci numbers.

TABLE IV. THE MULTIPLES OF 3 AND MULTIPLES OF 5 OF FIBONACCI NUMBERS (F_1 TO F_{30}).

n	F_n	Multiple of 3	Multiple of 5
4	3	3	
5	5		5
8	21	21	
10	55		55
12	144	144	
15	610		610
16	987	987	
20	6, 765	6, 765	6, 765
24	46, 368	46, 368	
25	75, 025		75, 025
28	317, 811	317, 811	
30	832, 040		832, 040

D. Fibonacci Numbers and Fermat's Theorem

Pierre de Fermat, a great mathematician from France and a lawyer by profession. Fermat points out the captivating feature about the following Fibonacci numbers; 1, 3, 8 and 120.
He says that "one more than product of any two of them is a perfect square"
That is,

$1 + (1 \text{ x } 3) = 4 = 2^2$ $\quad$ $1 + (3 \text{ x } 8) = 25 = 5^2$
$1 + (1 \text{ x } 8) = 9 = 3^2$ $\quad$ $1 + (3 \text{ x } 120) = 361 = 19^2$
$1 + (1 \text{ x } 120) = 121 = 11^2$ $\quad$ $1 + (8 \text{ x } 120) = 961 = 31^2$

Around 1969, there is a proof by Alan Baker and Harold Davenport of Trinity College, Cambridge – that if 1, 3, 8 and y having this property then the value of y will be 120.
Another interesting, is that; $1 = F_2$, $3 = F_4$, $8 = F_6$, and $120 = 4(2 \text{ x } 3 \text{ x } 5) = 4(F_3F_4F_5)$. The generalisation was established by V. Hoggatt, Jr. (one of the founder of Fibonacci Association) and G.E. Bergum of South Dakota State University after eight years Fermat observed the fascinating characteristic of the Fibonacci numbers [1](Koshy, 2001).

Theorem [1]
If the following numbers F_{2n} , F_{2n+2} , F_{2n+4} and $4F_{2n+1}F_{2n+2}F_{2n+3}$, have the attribute that one more than the product of any two of them is a perfect square.

Proof
According Thomas Koshy, the theorem was proved using Cassini's formula that; $1 + F_{2n}F_{2n+2} = F_{2n+1}^2$, $1 + F_{2n+1}F_{2n+3} = F_{2n+2}^2$, $1 + F_{2n+2}F_{2n+4} = F_{2n+3}^2$ and we have

$1 + F_{2n}(4F_{2n+1}F_{2n+2}F_{2n+3})$
$= 1 + 4(F_{2n+1}F_{2n+3})(F_{2n}F_{2n+2})$
$= 1 + 4(F_{2n+2}^2 + 1)(F_{2n+1}^2 - 1)$
$= 4(F_{2n+1}^2F_{2n+2}^2) - 4(F_{2n+2}^2 - F_{2n+1}^2) - 3$
$= 4(F_{2n+1}^2F_{2n+2}^2) - 4(F_{2n}F_{2n+3}) - 3$
$= 4(F_{2n+1}^2F_{2n+2}^2) - 4F_{2n+3}(F_{2n+2} - F_{2n+1}) - 3$
$= 4(F_{2n+1}^2F_{2n+2}^2) - 4(F_{2n+2}F_{2n+3}) + 4(F_{2n+1}F_{2n+3}) - 3$
$= 4(F_{2n+1}^2F_{2n+2}^2) - 4(F_{2n+2}F_{2n+3}) + 4(1 + F_{2n+2}^2) - 3$
$= 4(F_{2n+1}^2F_{2n+2}^2) - 4F_{2n+2}(F_{2n+3} - F_{2n+2}) + 1$
$= 4(F_{2n+1}^2F_{2n+2}^2) - 4(F_{2n+1}F_{2n+2}) + 1$
$= (2F_{2n+1}F_{2n+2} - 1)^2$

Similarly, $1 + F_{2n+2}(4F_{2n+1}F_{2n+2}F_{2n+3}) = (2F_{2n+2}^2 + 1)^2$ and $1 + F_{2n+4}(4F_{2n+1}F_{2n+2}F_{2n+3}) = (2F_{2n+2}F_{2n+3} + 1)^2$
As a result one can say, "one more than the product of any two of the number is a perfect square" is valid [1].

I. GOLDEN RATIO AND GOLDEN RECTANGLE

A. Golden Rectangle

A rectangle whose proportion of the sides is equal to the golden ratio is known as golden rectangle. The ratio of a golden rectangle is base on 1:1.6180 [13]. According to David Bergamini, the most visual satisfying of all forms of geometrics is golden rectangle [6]. It has been found aesthetically pleasing to human eye in the ways it is deliberately turning up in most of the art work [8] and architecture [13]. According to Stan Grist, reported that Mona Lisa's face was perfectly painted by a great Italian artist Leonardo da Vinci to fit golden rectangle, and used similar rectangles for the rest of his painting. Also, the front side of Parthenon building in Greece can be easily framed with golden rectangles [8].

Figure 19. **(a)** Mona Lisa portrait **(b)** Parthenon building

Source: *http://britton.disted.camosun.bc.ca/goldslide/jbgoldslide.html*

B. Golden Ratio and Human

i. Golden Ratio in Human Body

According to [14] statement, mentioned that the many parts of the human body has been proportioned according to golden ratio which is also taken as the basis in "Neufert" the most important reference book of modern day architect. However, it may not always be possible to use a ruler and find the ratio of all over people faces, because it applied to the idea of all human form of which the scientists and artists agreed. The researchers had confirmed that there are examples of golden ratio that exist in the average human body. The first example I will like to mention in the average human body is that when the distance between navel and foot is taken as 1 then the height of the human being is equivalent to 1.618. Some of the other examples of the golden ratio in the average human body are; if the distance between the wrist and the elbow is 1 then the distance between the finger tip and the elbow will be 1.618; when the distance between the jaw and the top of the head is 1 then the distance between the shoulder line and the top of the head will be 1.618. Also, if the distance from the shoulder line to the top of the head is taking as 1 then the distance from the navel to the top of the head will be equivalent to 1.618, similarly, when the distance between the knee and the end of the foot is 1 then the distance from the navel to the end of the foot will be 1.618. From the examples, one can simply agree and say that there is a close associate between golden ratio and human body and the golden mean or golden ratio can as well call the number of our physical body [1].

ii. Golden Ratio in Human Face

It had been reported by [14] that the divine ratio exists all over human face. For examples, the length of the face, 1.618 over the width of the face, 1; the distance between lips and where the eyebrow meets, 1.618 over the length of the nose, 1; the length of face, 1.618 over the distance from the tip of the jaw to where the eyebrow meets, 1. Also, the ratio of the length of mouth, 1.618 to the width of nose, 1 and the ratio of the width of nose, 1.618 to the distance between the nostril,1.

C. Golden Ratio in Great Pyramid

The Egyptians pyramids were known as the World's most beautiful and impressive monuments. The pyramids are fascinating to people because of the massive size and simple shape. The researchers have shown that the Greeks are not the first to use the visual balance of mysterious number φ = 1.618 in their architecture. In the dimension of many geometrical shapes, golden ratio has vital role in the two and three dimensions [15]. More than 4, 500 years ago, the ancient Egyptians base the design of the great pyramid of Giza on golden ratio. According to [1](Koshy, 2001) the great pyramid height is 484.4 feet, which is approximately 5813 inches, this give three consecutive Fibonacci sequence; 5, 8 and 13. A Greek historian, Herodotus reported that the Egyptian priests told him that the great pyramid proportions were chosen in a way that the "area of a square whose length of a side is the same as the height of the pyramid equal to the area of a triangular face" [6]. George Markowsky says that Herodotus statement means that the "ratio of the triangular face or slant height to half the length of the base of the great pyramid. To show this using figure 5.1, let the slant height be x, the height to be h and the base of the pyramid be $2y$. From the Herodotus formula we have; $h^2 = \frac{2y\,.\,x}{2}$ which then give the equation $h^2 = xy$ [6]. From Pythagorean Theorem we have $h^2 = x^2 - y^2$ [1].

Thus, $x^2 - y^2 = xy$ dividing through by y^2 we get $\left(\frac{x}{y}\right)^2 - 1 = \frac{x}{y}$ this can be writing as $\left(\frac{x}{y}\right)^2 - \frac{x}{y} - 1 = 0$ which satisfies the quadratic equation $\varphi^2 - \varphi - 1 = 0$ where $\varphi = \frac{x}{y}$.

According Thomas Koshy, the actual measurements of; the slant height is 188.4m, the height is 148.2m and the base of the great pyramid is 116.4m.

Hence, $\frac{x}{y} = \frac{188.4}{148.2} = 1.61855067010\ldots$ which is the value of the positive root of the golden ratio.

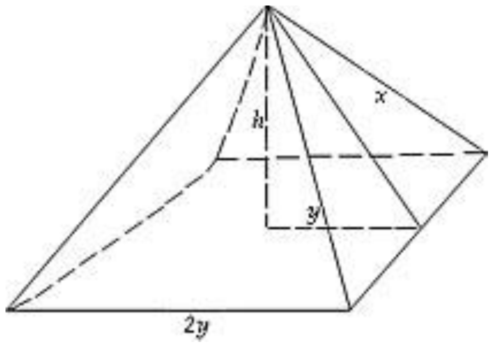

Figure 20. Pyramid

CONCLUSION AND EVALUATION

A. Conclusion and Recommendation

The main result of this report is the occurrence of Fibonacci numbers in nature and the relationship with golden ratio. Research had shown that the mysterious numbers (Fibonacci numbers) pop up in plants, fruits, flowers, human hand in which they have been fully discussed in this report. It was also shown that Fibonacci numbers can be generated mathematically through recursive relation formula, $F_n = F_{n-1} + F_{n-2}$ with initial condition, $F_1 = 1$ and $F_2 = 1$ for $n \geq 3$. The findings of this research shows that the numbers in the Fibonacci series were arranged in such a way that for every two odd numbers we have an even numbers, and the multiples of 3 are differ with equal interval as well as multiples of 5. This paper was able to take us towards enhancing the understanding of applications of Fibonacci sequence in mathematics and golden ratio in art and architecture. This study was undertaken to show the relationship between Fibonacci numbers and golden ratio through the drawing of a golden rectangle using Fibonacci numbers and 1:1.618, and the evaluation of $\frac{F_{n+1}}{F_n}$, when the value of *n* is getting larger and larger the result give the value of φ. Another intriguing finding that emerged in this study is the occurrence of Fibonacci numbers in Pascal's triangle and proves that golden ratio was implemented when constructing the Egyptians great pyramid of Giza.

We suggest an increase in future research in the area of golden ratio and human body and face. Also, it will be more interesting if an adequate research is done on the Egyptians pyramids, in which some authors have claimed that the constructions was based on golden ratio.

B. Limitation of the Research

The main significant limitations need to be acknowledged in this current study is the findings that human body and face are proportions according to golden ratio. The results of these findings are base on the literature of the scholars, since there is no practical measurement of human body and face in this research.

C. Critical Evaluation

During the course of this research it really appears that Fibonacci series and golden ratio have good mathematical relationship and computer scientists used some algorithms which are now classified as applications of Fibonacci numbers.

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