Abstract—In this paper, we discuss respectively the relationships between the feasible sets of the Weak product, the Cartesian product and the disjunctive product of uniform bi-hypergraphs and the feasible sets of the factors.

Keywords— hypergraph coloring; mixed hypergraph; feasible set; one-realization.

I. INTRODUCTION

A mixed hypergraph on a finite set $X$ is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where $\mathcal{C}$ and $\mathcal{D}$ families of subsets of $X$ are called $\mathcal{C}$-edges and $\mathcal{D}$-edges, respectively. A bi-edge is an edge which is both a $\mathcal{C}$-edge and a $\mathcal{D}$-edge. If $\mathcal{C} = \mathcal{D}$, then $\mathcal{H}$ is called a bi-hypergraph. If each edge has $r$ vertices, $\mathcal{H}$ is a $r$-uniform hypergraph if each edge has $r$ vertices. A sub-hypergraph $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D})$ of $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a spanning sub-hypergraph if $X' = X$, and a derived sub-hypergraph of $\mathcal{H}$ on $X'$, denoted by $\mathcal{H}[X']$, if $\mathcal{C}' = \{ C \subseteq C | C \subseteq X' \}$ and $\mathcal{D}' = \{ D \subseteq D | D \subseteq X' \}$.

Two mixed hypergraphs $\mathcal{H}_1 = (X_1, C_1, D_1)$ and $\mathcal{H}_2 = (X_2, C_2, D_2)$ are isomorphic if there exists a bijection $\phi$ from $X_1$ to $X_2$ that maps each $C$-edge of $\mathcal{C}_1$ onto a $C$-edge of $\mathcal{C}_2$ and maps each $D$-edge of $D_1$ onto a $D$-edge of $D_2$, and vice versa. The bijection $\phi$ is called an isomorphism from $\mathcal{H}_1$ to $\mathcal{H}_2$.

A proper $k$-coloring of $\mathcal{H}$ is a mapping from $X$ into a set of $k$ colors so that each $C$-edge has two vertices with a Common color and each $D$-edge has two vertices with Distinct colors. A strict $k$-coloring is a proper $k$-coloring using all of the $k$ colors, and a mixed hypergraph is $k$-colorable if it has a strict $k$-coloring. A coloring of $\mathcal{H}$ may be viewed as a partition of its vertex set, where the color classes are the sets of vertices assigned to the same color, so a strict $n$-coloring $c = \{C_1, C_2, \ldots, C_n\}$ of $\mathcal{H}$ means that $C_1, C_2, \ldots, C_n$ are the $n$ color classes under $c$. The set of all the values $k$ such that $\mathcal{H}$ has a strict $k$-coloring is called the feasible set of $\mathcal{H}$, denoted by $F(\mathcal{H})$. For each $k \in F(\mathcal{H})$, let $r_k$ denote the number of partitions of the vertex set. For a set $S$ of positive integers, we say that a mixed hypergraph $\mathcal{H}$ is a realization of $S$ if $F(\mathcal{H}) = S$. A mixed hypergraph $\mathcal{H}$ is a one-realization of $S$ if it is a realization of $S$ and $r_k = 1$ for each $k \in S$.

When one considers the colorings of a mixed hypergraph, it suffices to assume that each $C$-edge has at least three vertices. The study of the colorings of mixed hypergraphs has made a lot of progress since its inception ([6]). For more information, we would like refer readers to [3, 5, 7, 8].

Perhaps the most intriguing phenomenon of colorings of hypergraphs is that a mixed hypergraph can have gaps in its chromatic spectrum. We know that the feasible set of a classical hypergraph is an interval. Jiang et al. ([2]) proved that, for any finite set $S$ of integers greater than 1, there exists a mixed hypergraph $\mathcal{H}$ such that $F(\mathcal{H}) = S$, and Král ([4]) strengthened this result by showing that prescribing any positive integer $r_k$, there exists a mixed hypergraph which has precisely $r_k$ $k$-colorings for all $k \in S$. Recently, Bujtás and Tuza ([11]) gave the necessary and sufficient condition for a finite set $S$ of natural numbers being the feasible set of an $r$-uniform mixed hypergraph. Zhao et al. ([9]) proved that any vector $R = (r_0, r_1, \ldots, r_n)$ with $n > 2$ and $r_i \geq 0$, $i = 2, \ldots, n$ is the chromatic spectrum of some 3-uniform bi-hypergraph.

In this paper, we focus on the feasible sets of products of uniform bi-hypergraphs with relation to the feasible sets of the factors.

II. MAIN RESULTS

For any positive integer $n$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

We focus on the weak product, the Cartesian product and the disjunctive product of uniform bi-hypergraphs, respectively. We first discuss the feasible set of the weak product

Definition 2.1 For any two $r$-uniform bi-hypergraphs $\mathcal{H}_1 = (V_1, B_1)$ and $\mathcal{H}_2 = (V_2, B_2)$, the weak product of $\mathcal{H}_1$ and $\mathcal{H}_2$ is the $r$-uniform bi-hypergraph $\mathcal{H}_1 \times \mathcal{H}_2 = (V, B)$, where $V = V_1 \times V_2$ and

$$\{(x_1, y_1), \ldots, (x_r, y_r)\} \in B \iff \{x_1, \ldots, x_r\} \in B_1, \{y_1, \ldots, y_r\} \in B_2$$
Theorem 2.1 Let \( \mathcal{H}_1 = (V_1, B_1) \), \( \mathcal{H}_2 = (V_2, B_2) \) be two \( r \)-uniform bi-hypergraphs. Then
\[
\mathcal{F}(\mathcal{H}_1 \times \mathcal{H}_2) \supseteq \mathcal{F}(\mathcal{H}_1) \cup \mathcal{F}(\mathcal{H}_2).
\]

Proof For any \( t \in \mathcal{F}(\mathcal{H}_1) \) and any strict \( t \)-coloring \( c = \{C_1, C_2, \ldots, C_t\} \) of \( \mathcal{H}_1 \), write
\[
C'_t = \{(x, y) \mid x \in C_i, y \in V_2\}, i = 1, 2, \ldots, t \text{ and } c'_t = \{C'_1, C'_2, \ldots, C'_t\}.
\]
Note that for any \( B = \{(x_1, y_1), \ldots, (x_r, y_r)\} \in \mathcal{B} \), \( B_1 = \{x_1, \ldots, x_r\} \in \mathcal{B}_1 \). Hence, there are two vertices, say \( x_i, x_j \), such that \( c(x_i) = c(x_j) \), and two vertices, say \( x_k, x_m \), such that \( c(x_k) \neq c(x_m) \). Then
\[
c'_t((x_i, y_i)) = c'_t((x_j, y_j)) \quad \text{and} \quad c'_t((x_k, y_k)) \neq c'_t((x_m, y_m)),
\]
which implies that \( c' \) is a strict \( t \)-coloring of \( \mathcal{H}_1 \times \mathcal{H}_2 \). It follows that \( \mathcal{F}(\mathcal{H}_1) \subseteq \mathcal{F}(\mathcal{H}_1 \times \mathcal{H}_2) \). Similarly, we may have that \( \mathcal{F}(\mathcal{H}_2) \subseteq \mathcal{F}(\mathcal{H}_1 \times \mathcal{H}_2) \). Thus, the desired result follows.

The following result shows that the equality holds for some uniform bi-hypergraphs. We first construct the desired hypergraphs as follows.

For any set \( S = \{n_1, n_2, \ldots, n_s\} \) of positive integers with \( s \geq 2 \) and \( \min(S) \geq 2 \), let
\[
X_{n_1}, \ldots, n_s = \{(x_1, \ldots, x_s) \mid x_j \in [n_j], j \in [s]\}
\]
and
\[
B_{n_1}, \ldots, n_s = \{\{(x_1, \ldots, x_s), (y_1, \ldots, y_s), (z_1, \ldots, z_s)\} \mid \{x_j, y_j, z_j\} = 2, j \in [s]\}.
\]
Then \( (X_{n_1}, \ldots, n_s, B_{n_1}, \ldots, n_s) \) is a 3-uniform bi-hypergraph, denoted by \( \mathcal{H}_{n_1}, \ldots, n_s \). Moreover, we have the following result.

Lemma 2.2 ([9]) Let \( S = \{n_1, n_2, \ldots, n_s\} \) be a set of positive integers with \( s \geq 2 \) and \( \min(S) \geq 2 \). Then
\[
\mathcal{F}(\mathcal{H}_{n_1}, \ldots, n_s) = \{n_1, n_2, \ldots, n_s\}.
\]

Theorem 2.3 Let \( S_1 = \{n_1, \ldots, n_t\}, S_2 = \{m_1, \ldots, m_s\} \) be two sets of integers with \( s, t \geq 2 \) and \( \min(S_1), \min(S_2) \geq 2 \). Then there are two 3-uniform bi-hypergraphs, say \( \mathcal{H}_{S_1} \) and \( \mathcal{H}_{S_2} \), such that
\[
\mathcal{F}(\mathcal{H}_{S_1}) = S_1, \mathcal{F}(\mathcal{H}_{S_2}) = S_2 \quad \text{and} \quad \mathcal{F}(\mathcal{H}_{S_1} \times \mathcal{H}_{S_2}) = S_1 \cup S_2.
\]

Proof By Lemma 2.2, we have that
\[
\mathcal{F}(\mathcal{H}_{n_1}, \ldots, n_t) = S_1, \mathcal{F}(\mathcal{H}_{m_1}, \ldots, m_s) = S_2 \quad \text{and} \quad \mathcal{F}(\mathcal{H}_{n_1}, \ldots, n_t, m_1, \ldots, m_s) = S_1 \cup S_2.
\]

Let
\[
\phi : X_{n_1}, \ldots, n_t \times X_{m_1}, \ldots, m_s \rightarrow X_{n_1}, \ldots, n_t, m_1, \ldots, m_s
\]
\[
((x_1, \ldots, x_t), (y_1, \ldots, y_s)) \rightarrow (x_1, \ldots, x_t, y_1, \ldots, y_s)
\]
Then, it is not difficult to notice that \( \phi \) is an isomorphism from \( \mathcal{H}_{S_1} \times \mathcal{H}_{S_2} \) to \( \mathcal{H}_{n_1}, \ldots, n_t, m_1, \ldots, m_s \). Which follows that
\[
\mathcal{F}(\mathcal{H}_{S_1} \times \mathcal{H}_{S_2}) = S_1 \cup S_2.
\]

Next, we focus on the Cartesian product of bi-hypergraphs.

Definition 2.2 For any two \( r \)-uniform bi-hypergraphs \( \mathcal{H}_1 = (X_1, B_1), \mathcal{H}_2 = (X_2, B_2) \), the Cartesian product of \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) is the \( r \)-uniform bi-hypergraph \( \mathcal{H}_1 \square \mathcal{H}_2 = (X, B) \) with \( X = X_1 \times X_2 \) and
\[
\{(x_1, \ldots, x_r), (y_1, \ldots, y_r)\} \in B \text{ if and only if } \{x_1, \ldots, x_r\} \in B_1 \text{ and } \{y_1, \ldots, y_r\} \in B_2 \text{ and } x_1 = \cdots = x_r.
\]

Theorem 2.4 Let \( \mathcal{H}_1 = (X_1, B_1), \mathcal{H}_2 = (X_2, B_2) \) be two \( r \)-uniform bi-hypergraphs. Then
\[
\mathcal{F}(\mathcal{H}_1 \square \mathcal{H}_2) \subseteq \mathcal{F}(\mathcal{H}_1) \cap \mathcal{F}(\mathcal{H}_2).
\]

Proof For any strict coloring \( c = \{C_1, C_2, \ldots, C_k\} \) of \( \mathcal{H}_1 \square \mathcal{H}_2 \) and \( y \in X_2 \), let
\[
\begin{align*}
X_1 & \rightarrow X_1 \times \{y\} \\
x & \rightarrow (x, y).
\end{align*}
\]
Then \( \phi \) is an isomorphism from \( \mathcal{H}_1 \) to \( (\mathcal{H}_1 \square \mathcal{H}_2)[Y] \), where \( Y = X_1 \times \{y\} \). Note that is a strict \( k \)-coloring of, we have that \( \mathcal{H}_1 \) is \( k \)-colorable. Hence, \( \mathcal{F}(\mathcal{H}_1 \square \mathcal{H}_2) \subseteq \mathcal{F}(\mathcal{H}_1) \). Similarly, we may get that \( \mathcal{F}(\mathcal{H}_1 \square \mathcal{H}_2) \subseteq \mathcal{F}(\mathcal{H}_2) \), which implies that the desired result follows.

Lastly, we discuss the feasible set of the disjunctive product of bi-hypergraphs.

Definition 2.3 For any two \( r \)-uniform bi-hypergraphs \( \mathcal{H}_1 = (X_1, B_1), \mathcal{H}_2 = (X_2, B_2) \), the disjunctive product of \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) is the \( r \)-uniform bi-hypergraph \( \mathcal{H}_1 \ast \mathcal{H}_2 = (X, B) \), where \( X = X_1 \times X_2 \) and
\[
\{x_1, \ldots, x_r, y_1, \ldots, y_r\} \in B \text{ if and only if } \{x_1, \ldots, x_r\} \in B_1 \text{ or } \{y_1, \ldots, y_r\} \in B_2 \text{ or } x_1 = \cdots = x_r.
\]

Theorem 2.5 Let \( \mathcal{H}_1 = (X_1, B_1), \mathcal{H}_2 = (X_2, B_2) \) be two \( r \)-uniform bi-hypergraphs. Then
\[
\mathcal{F}(\mathcal{H}_1 \ast \mathcal{H}_2) \subseteq \mathcal{F}(\mathcal{H}_1) \cap \mathcal{F}(\mathcal{H}_2).
\]

Proof For any strict coloring \( c = \{C_1, C_2, \ldots, C_k\} \) of \( \mathcal{H}_1 \ast \mathcal{H}_2 \) and \( y \in X_2 \), let
\[
\begin{align*}
X_1 & \rightarrow X_1 \times \{y\} \\
x & \rightarrow (x, y).
\end{align*}
\]
Then \( \phi \) is an isomorphism from \( \mathcal{H}_1 \) to \( (\mathcal{H}_1 \ast \mathcal{H}_2)[Y] \), where \( Y = X_1 \times \{y\} \). Note that
\[ c' = \{ C_1 \cap Y, C_2 \cap Y, \ldots, C_k \cap Y \} \] is a strict \( k \)-coloring of \((H_1 \ast H_2)[Y]\), we have \( H_1 \) is \( k \)-colorable. Thus \( k \in \mathcal{F}(H_1) \) and we further have \( \mathcal{F}(H_1 \ast H_2) \subseteq \mathcal{F}(H_1) \).

Similarly, \( \mathcal{F}(H_1 \ast H_2) \subseteq \mathcal{F}(H_2) \). That follows that \( \mathcal{F}(H_1 \ast H_2) \subseteq \mathcal{F}(H_1) \cap \mathcal{F}(H_2) \).

### III. CONCLUSIONS

In this paper, we discuss the feasible sets of several products of uniform bi-hypergraphs with relation to the feasible sets of the factors. Precisely, we prove that the feasible set of the Cartesian product or the disjunctive product of two \( r \)-uniform bi-hypergraphs is a subset of the intersection of the feasible sets of the factors, and the feasible set of the weak product of two \( r \)-uniform bi-hypergraphs contains the union of the feasible sets of the factors.

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### REFERENCES


