Two-Grid Finite Volume Element Method for Second-Order Quasi-Linear Parabolic Problems

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Abstract--- **In this paper, we are concerned with the two-grid finite volume element methods to the second-order quasi-linear parabolic problems. Two-grid finite volume element methods are based on two linear conforming finite element spaces on one coarse grid and one fine grid. Here, it is proved that the coarse grid can be much coarser than the fine grid. With the proposed techniques, solving the nonlinear problems is reduced to solving a linear problem on the fine space. Convergence estimates are derived to justify the efficiency of the proposed two-grid algorithms. A numerical experiment confirms some results of theoretical analysis.**

Keywords--- two-grid finite volume element method, second order quasi-linear parabolic problem, second order partial differential equation

I. INTRODUCTION

The finite volume element methods (FVEMs) are a discretization technique for the partial differential equations arising from physical conservation laws including mass, energy.

The finite volume element methods are the special cases of generalized difference methods.[1-6]

The finite volume element methods discretize the integral form of conservation law of differential equation by choosing linear or bilinear finite element space as trial space.

They have the simplicity of finite difference methods and the accuracy of finite element methods and have been widely used in computational fluid mechanics because they keep the conservation law of mass or energy.

Cai and Steve Mccormic [1] had presented finite volume element method for diffusion equations on composite grids and provided the error estimates which were relatively complicated.

Afterwards, they gave simple theoretical analysis for diffusion equations on general triangulations.

However, it was constrained to special choosing of control volumes.

Li Qian and his colleagues [7,8] also had a lot of contributions to the studies of finite volume element methods. Plexousakis and Zouraris [9] derived a class of high order finite volume element methods for solving one dimensional elliptic equation. Cai, Douglas and Park [10] constructed a high order finite volume element method by mixed variational principle. They presented a ways to derive high order finite volume element method over rectangular meshes.

Mishev [11] has considered the FVEM in the linear conforming finite element space and has established the error estimate in the H^1-norm.

Wu and Li [12] have obtained the H^1 superconvergence and L^p error estimates between the solution of the FVEM and that of the finite element method.

Li [13] has considered the finite volume element method for a nonlinear elliptic problem and obtained the error estimate in the H^1 norm.

Xu [14-16] has studied the two-grid finite element method based on two finite element spaces on one coarse and one fine grid for non-symmetric and nonlinear elliptic problems.

Late on, Xu, Zhou [17] for eigenvalue problems, Axelsson and Layton [18] for nonlinear elliptic problems, Dawson, Wheeler and Woodward [19] for finite difference scheme for nonlinear parabolic equations, Layton and Lenferink [20] and Utnes [21] for Navier-Stokes equations, Marion and Xu [22] for evolution equations have considered the two grid method.

C.J. Bi and V. Ginting [23] have studied two grid finite volume element discretization techniques for the non-selfadjoint and definite linear elliptic problems and the nonlinear elliptic problems based on two linear conforming finite element spaces V_H and V_h with grid size H and h ($H >> h$).

In this paper, we consider two-grid finite volume element method for two-dimensional quasi-linear parabolic equation based on two linear conforming finite element spaces with coarse grid and dense grid.

The rest of the article is organized as follows: In Section 2, we describe the FVEM for the quasi-linear parabolic equation and algorithm of two-grid finite volume element scheme. Section 3 contains the error analysis by two-grid finite volume element method for quasi-linear parabolic equation.

II. -TWO GRID FINITE VOLUME ELEMENT SCHEME

We consider the initial-boundary value problem of quasi-linear parabolic equations

$$\frac{\partial u}{\partial t} - div(\alpha(x,t,u)\nabla u) = f(x,t,u), \qquad (x,t) \in \Omega \times (0,T] \text{ , (1.1)}$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T] \text{ , (1.2)}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega \text{ , (1.3)}$$

where $\Omega \subset R^2$ is a bounded-closed, convex domain.

We assume that $\alpha(x,t,u)$, $f(x,t,u)$ are smooth functions on $\Omega \times (0,T] \times R$ and equations (1.1) - (1.3) have the only unique solution $u \in H_0^1(\Omega)$ on the $\Omega \times (0,T]$.

Writing the variational equations of equations (1.1) – (1.3),

$$\int_\Omega \frac{\partial u}{\partial t} v dx + \int_\Omega (\alpha(x,t,u)\nabla u)\cdot\nabla v dx = \int_\Omega f(x,t,u) v dx, \qquad \forall v \in H_0^1(\Omega).$$

The weak formulation of (1.1) – (1.3) is

$$(\frac{\partial u}{\partial t}, v) + a(u\,; u, v) = (f(x,t,u), v) \ , \quad \forall v \in H_0^1(\Omega) \text{, (1.4)}$$

where $(\ \cdot, \cdot\ )$ denotes the $L^2(\Omega)$ –inner product and the bilinear $a(\cdot\,,\cdot)$ is defined by $a(u, v) = \int_\Omega (\alpha(x,t,u)\nabla u)\cdot\nabla v dx$, $\forall u, v \in H_0^1(\Omega)$ and $a(w\,; u, v) = (\alpha(x,t,w)\nabla u, \nabla v)$.

Let T_h be a quasi – uniform triangulation of Ω with $h = \max\{h_k\}$, where h_k is the diameter of the element triangle $k \in T_h$.

We consider a finite element discretization of (1.4) in the standard conforming finite element space V_h of piecewise linear functions, defined on the triangulation T_h,

$$V_h = \{v \mid v \in C(\Omega),\ v|_{\mathrm{k}} \text{ is linear},\ v|_{\partial\Omega} = 0, \quad \forall k \in T_h\}.$$

In order to describe the FVEM for solving equation (1.1) – (1.3), we construct a dual partition T_h^* based upon the original triangulation T_h whose elements are called the control volumes.

We construct the control volume as follows; Let z_k be the barycenter of $K \in T_h$. We connect z_k with line segments to the midpoints of the edges of K, thus partitioning K into three quadrilaterals K_z, $z \in Z_h(K)$, where $Z_h(K)$ are the set of vertices of K.

Then with each vertex $z \in Z_h = \bigcup_{K \in T_h} Z_h(K)$ we associate a control volume w_z, which consists of the union of the subregions K_z, sharing the vertex z. Thus we obtain a group of control volumes covering the domain Ω, which is called the dual partition T_h^* of the triangulation T_h.

We denote the set of interior vertices of Z_h by Z_h^0.

We call the partition T_h^* regular or quasi-uniform, if there exists a positive constant $C > 0$ such that $C^{-1}h^2 \le meas(w_z) \le Ch^2$, $\forall w_z \in T_h^*$.

We formulate the FVEM for the equation (1.1)) – (1.3) as follows.

Given the vertex $z \in Z_h$, integrating equation (1.1)) – (1.3) over the associated control volume w_z and applying Green's formula, we obtain

$\int_{w_z} \frac{\partial u}{\partial t} dx + \int_{\partial w_z} (\alpha(x,t,u)\nabla u)\cdot n ds = \int_{w_z} f(x,t,u)\, dx$, (1.5) where n denotes the unit outer-normal vector on ∂w_z.

The semi-discrete FVE approximation solution of (1.1) – (1.3) is defined as a $u_h(x,t) \in V_h \ (0 \le t \le T)$, such that

$$\int_{w_h} \frac{\partial u_h}{\partial t} dx + \int_{\partial w_h} (\alpha(x,t,u_h)\nabla u_h)\cdot n ds = \int_{w_h} f(x,t,u_h)\, dx. \quad (1.6)$$

Now the interpolation operator $\Pi_h^* : V_h \to V_h^*$ is defined by

$$\Pi_h^* v_h = \sum_{z \in Z_h^0} v_h(z)\chi_z \quad , \quad (1.7)$$

where $V_h^* = \{v \mid v \in L^2(\Omega),\ v|_{w_z} \text{ is constant for all } w_z \in T_h^*;\ v|_{w_z} = 0, \text{ if } z \in \partial\Omega\}$ and χ_z is the characteristic function of the control volume w_z.

The semi-discrete FVEM (1.6) can be rewritten in a variational form

$$\begin{cases} (\frac{\partial u_h}{\partial t}, \Pi_h^* v_h) + a_h(u_h; u_h, \Pi_h^* v_h) = (f(x,t,u_h), \Pi_h^* v_h) \\ u_h(x,0) = u_{0h}(x) \end{cases}, \quad (1.8)$$

where for any $u_h, v_h, w_h \in V_h$ the bilinear form $a_h(\cdot\,;\cdot, \Pi_h^*\cdot)$ is defined by

$$a_h(w_h; u_h, \Pi_h^* v_h) = -\sum_{z \in Z_h^0} \int_{\partial w_z} (\alpha(\cdot, w_h)\nabla u_h)\cdot n \Pi_h^* v_h ds = -\sum_{z \in Z_h^0} v_h(z) \int_{\partial w_h} (\alpha(x,t,w_h)\nabla u_h)\cdot n ds \quad (1.9)$$

Next we partition off the interval $(0, T]$ in order to derive full discrete FVE scheme.

Let τ be the time step size. $t^k = k\tau$ and $u_h(t_k) = u_h^k$.

Writing the full discrete FVE scheme for equation (1.1) – (1.3),

$$\begin{cases} (\partial_t u_h^k, \Pi_h^* v_h^k) + a_h(u_h^k; u_h^k, \Pi_h^* v_h^k) = f((x,t^k,u_h^k), \Pi_h^* v_h^k), & \forall v_h^k \in V_h \\ u_h^0 = u_{0h}(x), & x \in \Omega \end{cases} \quad (1.10)$$

where $\partial_t u_h^k = \frac{u_h^k - u_h^{k-1}}{\tau}$.

We shall present the two-grid finite volume element algorithm for the equation (1.10) based on two finite element spaces.

The two-grid method is to reduce the quasi-linear problem on a fine grid into a linear problem by solving a quasi-linear problem on a coarse grid.

Let T_H and T_h be two quasi-uniform triangulations of Ω with two different mesh size H, $h \ (H > h)$.

T_H, T_h will be called the coarse grid, the fine grid, respectively.

V_H, V_h are the corresponding finite element spaces.

The two-grid finite volume element algorithm of the equation (1.10) is as follows.

① Find $u_H^k \in V_H (k = 1,2,...)$ such that

$$\begin{cases} (\partial_t u_H^k, \Pi_H^* v_H^k) + a_H(u_H^k; u_H^k, \Pi_H^* v_H^k) = f((x,t^k,u_H^k), \Pi_H^* v_H^k), & \forall v_H^k \in V_H \\ u_H^0 = u_{0H}(x), & x \in \Omega \end{cases} \quad (1.11)$$

on the coarse grid T_H.

② Find $u_h^k \in V_h (k = 1, 2,...)$ such that

$$\begin{cases} (\partial_t u_h^k, \Pi_h^* v_h^k) + a_h(u_H^k; u_h^k, \Pi_h^* v_h^k) = f((x,t^k,u_H^k), \Pi_h^* v_h^k), & \forall v_h^k \in V_h \\ u_h^0 = u_{0h}(x), & x \in \Omega \end{cases} \quad (1.12)$$

on the fine grid T_h.

This approximate solution u_h^k is called the approximate solution of the two-grid finite volume element method of equations (1.1) – (1.3).

III. ERROR ANALYSIS OF APPROXIMATE SOLUTION

In this section, we shall present the error estimate for the two-grid finite volume element method.

To describe error estimates, we first define some discrete norms on V_h.

$$|u_h|_{0,h}^2 = (u_h, u_h)_{0,h}, \quad (u_h, u_h)_{0,h} = \sum_{x_i \in Z_h} meas(V_i) u_{h_i} u_{h_i} = (\Pi_h^* u_h, \Pi_h^* u_h)$$

$$|u_h|_{1,h}^2 = \sum_{x_i \in Z_h} \sum_{x_i \in Z_h} meas(V_i)((u_{h_i} - u_{hj})/d_{ij})^2, \quad \|u_h\|_{1,h}^2 = |u_h|_{0,h}^2 + |u_h|_{1,h}^2, \quad |||u_h|||_0^2 = (u_h, \Pi_h^* u_h) \qquad (2.1)$$

We introduce the following bilinear forms

$$a(w_h; u_h, v_h) = \int_\Omega (\alpha(w_h)\nabla u_h)\cdot\nabla v_h dx, \qquad a_c(w_h; u_h, v_h) = \int_\Omega (\overline{\alpha(w_h)}\nabla u_h)\cdot\nabla v_h dx,$$

$$a_h(w_h; u_h, \Pi_h^* v_h) = -\sum_{z\in N_z}\int_{\partial w_z} (\alpha(\cdot, w_h)\nabla u_h)\cdot n\Pi_h^* v_h ds,$$

$$a_{h,c}(w_h; u_h, \Pi_h^* v_h) = -\sum_{z\in N_z}\int_{\partial w_z} (\overline{\alpha(\cdot, w_h)}\nabla u_h)\cdot n\Pi_h^* v_h ds,$$

where $\overline{\alpha(w_h)}\big|_k = \dfrac{1}{meas(K)}\int_\Omega \alpha(\cdot, w_h)dx$ and $d_{ij} = d(x_i, x_j)$ is the distance between x_i and x_j.

For $u, w \in H_0^1(\Omega)$ we assume that $a_h(w; u, \Pi_h^* u) \ge C_2 \|u\|_1^2$.

Lemma 1. For any arbitrary $u_h, v_h \in V_h$, we have

$$a_h(u_h; u_h, v_h) - a_h(u_h; u_h, \Pi_h^* v_h) =$$
$$= \sum_{K\in T_h}\int_{\partial K} (\alpha(x,t,u_h)\nabla u_h + \beta(x,t,u_h))\cdot n(v_h - \Pi_h^* v_h)ds - \qquad (2.2)$$
$$- \sum_{K\in T_h}\int_K (\alpha(x,t,u_h)\nabla u_h + \beta(x,t,u_h))\cdot(v_h - \Pi_h^* v_h)dx$$

Proof. Using the formula of Green,

$$\sum_{k\in T_h}(div(\alpha(x,t,u_h)\nabla u_h, v_h) = \sum_{k\in T}\int_k div(\alpha(x,t,u_h)\nabla u_h)v_h dx =$$
$$= \sum_{k\in T}\int_{\partial k} (\alpha(x,t,u_h)\nabla u_h)\cdot n v_h ds - a_h(u_h; u_h, v_h)$$

Moreover

$$\sum_{k\in T_h}(div(\alpha(x,t,u_h)\nabla u_h), \Pi_h^* v_h) = \sum_{k\in T}\sum_{x_j\in Z_h}(div(\alpha(x,t,u_h)\nabla u_h), \Pi_h^* v_h)_{k\cap w_{x_j}} =$$
$$= \sum_{k\in T}\int_{\partial k} (\alpha(x,t,u_h)\nabla u_h)\cdot n\Pi_h^* v_h ds - \sum_{x_j\in Z_h}\int_{\partial w_{x_j}} ((\alpha(x,t,u_h)\nabla u_h)\cdot n\Pi_h^* v_h ds =$$
$$= \sum_{k\in T}\int_{\partial k} (\alpha(x,t,u_h)\nabla u_h)\cdot n\Pi_h^* v_h ds - a_h(u_h; u_h, \Pi_h^* v_h)$$

From two expressions above

$$a_h(u_h; u_h, v_h) - a_h(u_h; u_h, \Pi_h^* v_h) =$$
$$= \sum_{k\in T_h}\int_{\partial k} (\alpha(x,t,u_h)\nabla u_h)\cdot n(v_h - \Pi_h^* v_h)ds - \sum_{k\in T_h}\int_k (\alpha(x,t,u_h)\nabla u_h)\cdot(v_h - \Pi_h^* v_h)dx.$$

The following lemma is proved in [3].

Lemma 2. For any arbitrary $u_h \in V_h$, there exist a positive constant C_0, C_1 such that

$$C_0|u_h|_{0,h} \le \|u_h\|_0 \le C_1|u_h|_{0,h}$$
$$C_0|||u_h|||_0 \le \|u_h\|_0 \le C_1|||u_h|||_0 \quad (2.3)$$
$$C_0\|u_h\|_{1,h} \le \|u_h\|_1 \le C_1\|u_h\|_{1,h}$$

Lemma 3. For any arbitrary $u_h \in V_h$, there exist a positive constant C_1, C_2 such that

$$\|\Pi_h^* u_h\| \le C_1\|u_h\|$$
$$a_h(w_h; u_h, \Pi_h^* u_h) \ge C_2\|u_h\|_1^2 \qquad \forall u_h, w_h \in V_h \quad (2.4)$$

proof.

From inequality $\|\Pi_h^* u_h\|^2 \le (\Pi_h^* u_h, \Pi_h^* u_h) = \sum_{x_i \in N_h} u_{h_i} u_{h_i} \cdot meas(w_i)$, the first inequality is established.

The second inequality is established from the given condition.

Let $p_h : H^2(\Omega) \to V_h$ be operator defined by equation $a_h(w; u - p_h u, \Pi_h^* v_h) = 0, \ \forall v_h \in V_h$.

We call elliptic projection of $u \in H^2(\Omega) \cap H_0^1(\Omega)$ operator $p_h : H^2(\Omega) \to V_h$.

Now we give the error of the approximate solution of the finite volume element method for the equations (1.1) – (1.3).

Theorem 1. Let u and u_h^k be the solution of equation (1.1) – (1.3) and (1.10), respectively.

Assume that $f(x,t,u)$ satisfies the inequality

$$|f(x,t,w) - f(x,t,v)| \le C|w - v|, \quad \forall w, v \in R.$$

Then, there exist a positive constant $\overline{C}$ such that

$$\|u_h^k - u^k\| \le \overline{C}(\|u_0 - u_{0h}\| + h^2\|u_0\|_3 + h^2\|u^k\|_3 + \tau\|u_{tth}^k\|). \quad (2.5)$$

proof. By ellipse projection operator p_h

$$u_h^k - u^k = (u_h^k - p_h u^k) + (p_h u^k - u) = \xi^k + \eta^k \quad (2.6)$$

From the equation (1.10) and the elliptic operator p_h,

$$(\partial_t \xi^k, \Pi_h^* v_h) + a_h(u_h^k; \xi^k, \Pi_h^* v_h) =$$
$$= (\partial_t p_h u^k, \Pi_h^* v_h) - (f(\cdot, u_h^k), \Pi_h^* v_h) + a_h(u_h^k; p_h u^k, \Pi_h^* v_h) =$$
$$= (\partial_t p_h u^k, \Pi_h^* v_h) - (f(\cdot, u_h^k), \Pi_h^* v_h) + [a_h(u_h^k; u^k, \Pi_h^* v_h) - a_h(u^k; u^k, \Pi_h^* v_h)] -$$
$$- a_h(u^k; u^k, \Pi_h^* v_h) =$$
$$= (\partial_t p_h u^k - u_t^k, \Pi_h^* v_h) + [(f(\cdot, u^k) - f(\cdot, u_h^k), \Pi_h^* v_h)] -$$
$$-[a_h(u_h^k; u^k, \Pi_h^* v_h) - a_h(u^k; u^k, \Pi_h^* v_h)].$$
$$\forall v_h \in V_h$$

Choosing $v_h = \xi^k$,

$$(\partial_t \xi^k, \Pi_h^*(\xi^k)) + a_h(u_h^k; \xi^k, \Pi_h^* \xi^k) =$$
$$= (\partial_t p_h u^k - u_t^k, \Pi_h^* \xi^k) + [(f(\cdot, u^k) - f(\cdot, u_h^k), \Pi_h^* \xi^k)] +$$
$$-[a_h(u_h^k; u^k, \Pi_h^* \xi^k) - a_h(u^k; u^k, \Pi_h^* \xi^k)]$$

By Lemma 3,

$$\frac{1}{\tau}(\xi^k - \xi^{k-1}, \Pi_h^*\xi^k) \le (\partial_t p_h u^k - u_t^k, \Pi_h^*\xi^k) + [(f(\cdot, u^k) - f(\cdot, u_h^k), \Pi_h^*\xi^k)] + \\ -[a_h(u_h^k; u^k, \Pi_h^*\xi^k) - a_h(u^k; u^k, \Pi_h^*\xi^k)]$$

Rewriting,

$$\frac{1}{\tau}(\xi^k - \xi^{k-1}, \Pi_h^*(\xi^k + \xi^{k-1})) \le (\partial_t p_h u^k - u_t^k, \Pi_h^*(\xi^k + \xi^{k-1})) + \\ +[(f(\cdot, u^k) - f(\cdot, u_h^k), \Pi_h^*(\xi^k + \xi^{k-1}))] + \\ -[a_h(u_h^k; u^k, \Pi_h^*(\xi^k + \xi^{k-1})) - a_h(u^k; u^k, \Pi_h^*(\xi^k + \xi^{k-1}))]$$

Then

$$\frac{1}{\tau}(\left\|\xi^k\right\|^2 - \left\|\xi^{k-1}\right\|^2) \le \left\|\partial_t p_h u^k - u_t^k\right\| \cdot \left\|\Pi_h^*(\xi^k + \xi^{k-1})\right\| + \\ + \left\|(f(\cdot, u^k) - f(\cdot, u_h^k)\right\| \cdot \left\|\Pi_h^*(\xi^k + \xi^{k-1})\right\| + c(\left\|\xi^k\right\| + \left\|\eta^k\right\|) \cdot \left\|\Pi_h^*(\xi^k + \xi^{k-1})\right\| \le \\ \le (\left\|\partial_t p_h u^k - u_t^k\right\| + \left\|(f(\cdot, u^k) - f(\cdot, u_h^k)\right\| + c(\left\|\xi^k\right\| + \left\|\eta^k\right\|)) \cdot \left\|\xi^k + \xi^{k-1}\right\|$$

(2.7)

By Lemma 2

$$\frac{1}{\tau}(\left\|\xi^k\right\|^2 - \left\|\xi^{k-1}\right\|^2) \le (\left\|\partial_t p_h u^k - u_t^k\right\| + \left\|(f(\cdot, u^k) - f(\cdot, u_h^k)\right\| + c(\left\|\xi^k\right\| + \left\|\eta^k\right\|)) \cdot (\left\|\xi^k\right\| + \left\|\xi^{k-1}\right\|)$$ Rewriting,

$$\frac{1}{\tau}(\left\|\xi^k\right\| - \left\|\xi^{k-1}\right\|) \le (\left\|\partial_t p_h u^k - u_t^k\right\| + \left\|(f(\cdot, u^k) - f(\cdot, u_h^k)\right\| + c(\left\|\xi^k\right\| + \left\|\eta^k\right\|))$$

Using $\left\|(f(\cdot, u^k) - f(\cdot, u_h^k)\right\| \le c_1(\left\|\xi^k\right\| + \left\|\eta^k\right\|)$ [4],

$\left\|\xi^k\right\| \le \left\|\xi^{k-1}\right\| + \tau(\left\|\partial_t p_h u^k - u_t^k\right\| + c_2(\left\|\xi^k\right\| + \left\|\eta^k\right\|))$. (2.8)

From (2.8) $\left\|\xi^k\right\| \le \frac{1}{1 - c_2\tau}\left\|\xi^{k-1}\right\| + \frac{\tau}{1 - c_2\tau}(\left\|\partial_t p_h u^k - u_t^k\right\| + c_2\left\|\eta^k\right\|)$.

Then

$$\left\|\xi^k\right\| \le \frac{1}{1 - c_2\tau}\left\|\xi^{k-1}\right\| + \left(\frac{1 + c_2\tau}{1 - c_2\tau}\right)(\left\|\partial_t p_h u^k - u_t^k\right\| + c_2\left\|\eta^k\right\|) \le$$

$$\le \frac{1}{1 - c_2\tau}\left\|\xi^{k-1}\right\| + \left(\frac{1 + c_2\tau}{1 - c_2\tau}\right)^n (\left\|\partial_t p_h u^k - u_t^k\right\| + c_2\left\|\eta^k\right\|). \quad (2.9)$$

Note that $\tau = \frac{T}{N}$, then $\frac{1}{1 - c_2\tau} = \frac{1}{1 - c_2\frac{T}{N}}$ $\left(\frac{1 + c_2\tau}{1 - c_2\tau}\right)^n \le \left(\frac{1 + c_2\tau}{1 - c_2\tau}\right)^N = \left(\frac{N + c_2T}{N - c_2T}\right)^N$.

Therefore, $\frac{1}{1 - c_2\tau} \to 1, \left(\frac{N + c_2T}{N - c_2T}\right)^N \to e^{c_2T} < \infty$ ($N \to \infty$).

By the (2.9) we have

$$\left\|\xi^k\right\| \le C\left\|\xi^0\right\| + \sum_{j=1}^{k}(\left\|\partial_t p_h u^k - u_t^k\right\| + c_2\left\|\eta^k\right\|) \quad (2.10)$$

In the (2.10)

$$\|\xi^0\| = \|p_h u_0 - u_{0h}\| \le \|u_0 - u_{0h}\| + \|\eta^0\| \le \|u_0 - u_{0h}\| + ch^2\|u_0\|_3$$

$$\|\eta^k\| \le ch^2 \|u^k\|_3 \qquad (2.11)$$

$$\|\partial_t p_h u^k - u_t^k\| \le \tau \|u_{tth}^k\|$$

From (2.11)

$$\|\xi^k\| \le C(\|u_0 - u_{0h}\| + ch^2\|u_0\|_3 + ch^2\|u^k\|_3 + \tau\|u_{tth}^k\|) \quad (2.12)$$

Combining (2.6) – (2.12), we have

$$\|u_h^k - u^k\| \le \|u_h^k - p_h u^k\| + \|p_h u^k - u^k\| \le$$

$$\le C(\|u_0 - u_{0h}\| + h^2\|u_0\|_3 + h^2\|u^k\|_3 + \tau\|u_{tth}^k\|) + C' h^2 \|u^k\|_3 \le$$

$$\le \overline{C}(\|u_0 - u_{0h}\| + h^2\|u_0\|_3 + h^2\|u^k\|_3 + \tau\|u_{tth}^k\|)$$

Next, we give the error of the approximate solution of the two-grid finite volume element scheme (1.11) - (1.12) for the equations (1.1) – (1.3).

Theorem 2. Let u and u_h^k be the solution (1.1) – (1.3) and (1.11) – (1.12), respectively.

Then for h, H sufficiently small ,there exists a positive constant c such that

$$\|u^k - u_h^k\|_1 \le c((h + H^2)\|u_h^k\|_2 + \tau\|u_{tth}^k\|) \quad (2.13)$$

proof. By the equation (1.11) – (1.12) and the elliptic projection operator p_h,

$$a_{h,c}(u_H^k; u_h^k - p_h u^k, \Pi_h^* v_h) = [a_{h,c}(u_H^k; u_h^k, \Pi_h^* v_h) - a_h(u_h^k; p_h u^k, \Pi_h^* v_h)] +$$

$$+ [a_h(u_h^k; p_h u^k, \Pi_h^* v_h) - a_{h,c}(u_H^k; p_h u^k, \Pi_h^* v_h)] =$$

$$= [a_{h,c}(u_H^k; u_h^k, \Pi_h^* v_h) - a_h(u_h^k; u^k, \Pi_h^* v_h)] +$$

$$+ [a_h(u_h^k; p_h u^k, \Pi_h^* v_h) - a_{h,c}(u_H^k; p_h u^k, \Pi_h^* v_h)] =$$

$$= (\partial_t u_h^k, \Pi_h^* v_h) - (f(\cdot, u_H^k), \Pi_h^* v_h) - (u_{th}^k, \Pi_h^* v_h) + (f(\cdot, u^k), \Pi_h^* v_h) +$$

$$+ [a_h(u_h^k; p_h u^k, \Pi_h^* v_h) - a_{h,c}(u_H^k; p_h u^k, \Pi_h^* v_h)] =$$

$$= (\partial_t u_h^k - u_{th}^k, \Pi_h^* v_h) + (f(\cdot, u^k) - f(\cdot, u_H^k), \Pi_h^* v_h) +$$

$$+ [a_h(u_h^k; p_h u^k, \Pi_h^* v_h) - a_{h,c}(u_H^k; p_h u^k, \Pi_h^* v_h)] =$$

$$= Q_1 + Q_2 + Q_3$$

$$|Q_1| = |(\partial_t u_h^k - u_{t,h}^k, \Pi_h^* v_h)| \le \|\partial_t u_h^k - u_{t,h}^k\| \cdot \|\Pi_h^* v_h\| \le c\tau \|u_{tth}^k\| \cdot \|v_h\|$$

$$|Q_2| = |(f(\cdot, u^k) - f(\cdot, u_H^k),\ \Pi_h^* v_h)| \le |(f(\cdot, u^k) - f(\cdot, u_H^k),\ \Pi_H^*(\Pi v_h))| +$$

$$|(f(\cdot, u^k) - f(\cdot, u_H^k),\ \Pi_h^*(v_h - \Pi_H v_h))| + |(f(\cdot, u^k) - f(\cdot, u_H^k),\ \Pi_h^*(\Pi_H v_h) - \Pi_H^*(\Pi_h v_h))| \le$$

$$\le cH^2 \|\Pi_H v_h\|_1 + cH \|u^k - u_H^k\|_1 \cdot \|v_h\|_1 + c(h + H)\|u^k - u_H^k\|_1 \cdot C \le cH^2 \|v_h\|_1$$

$$|Q_3| = |[a_h(u_h^k;\, p_h u^k, C) - a_{h,c}(u_H^k;\, p_h u^k, \Pi_h^* v_h)]| = \left| -\sum_z \int_{\partial V_z} (\alpha(\cdot, u_h^k)\nabla p_h u^k)\cdot n\, \Pi_h^* v_h ds + \sum_z \int_{\partial V_z} (\overline{\alpha(\cdot, u_H^k)}\nabla p_h u^k)\cdot n\, \Pi_h^* v_h ds \right| \le$$

$$\le \sum_z \int_{\partial V_z} (|\alpha(\cdot, u_h^k) - \overline{\alpha(\cdot, u_H^k)}|\nabla p_h u^k)\cdot n\, \Pi_h^* v_h ds \le c\|u_h^k - u_H^k\| \cdot \|p_h u_h^k\|_1 \cdot \|v_h\|_1 \le c(\|u_h^k - u^k\| + \|u^k - u_H^k\|)\cdot \|u_h^k\|_1 \cdot \|v_h\|_1 \le$$

$$\le c(h^2 + H^2)\|u_h^k\|_2 \cdot \|v_h\|_1$$

Therefore

$$a_{h,c}(u_H^k; u_h^k - p_h u^k, \Pi_h^* v_h) \le c(\tau\|u_{tth}^k\| \cdot \|\Pi_h^* v_h\| + H^2 \cdot \|\Pi_h^* v_h\| + (h^2 + H^2)\|u_h^k\|_2 \cdot \|\Pi_h^* v_h\| \le$$

$$\le c(\tau\|u_{tth}^k\| \cdot \|v_h\|_1 + H^2 \cdot \|v_h\|_1 + (h^2 + H^2)\|u_h^k\|_2 \cdot \|v_h\|_1$$

Choosing $v_h = u_h^k - p_h u^k$, from expression above

$$c\|u_h^k - p_h u^k\|_1^2 \le a_{h,c}(u_H^k; u_h^k - p_h u^k,\ \Pi_h^*(u_h^k - p_h u^k)) \le$$

$$\le c(H^2 + (h^2 + H^2)\|u_h^k\|_2 + \tau\|u_{tth}^k\|) \cdot \|u_h^k - p_h u^k\|$$

Then $\|u_h^k - p_h u^k\|_1 \le c(H^2 + (h^2 + H^2)\|u_h^k\|_2 + \tau\|u_{tth}^k\|)$

Therefore

$$\|u^k - u_h^k\|_1 \le \|u^k - p_h u^k\|_1 + \|p_h u^k - u_h^k\|_1 \le ch \cdot \|u^k\|_2 + c(H^2 + (h^2 + H^2)\|u_h^k\|_2 + \tau\|u_{tth}^k\|) \le$$

$$\le c((h + H^2)\|u_h^k\|_2 + \tau\|u_{tth}^k\|)$$

IV. CONCLUSION

In this paper, we have studied present a full discrete scheme and algorithm of the two – grid finite volume element method for a quasi - linear parabolic equation.

We have estimated the error of the approximate solution by the two – grid finite volume element method.

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